Relaxation and the Dynamics of Staircase Formation: Some Physics Beyond the Color VGs

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U.C.S.D.

<u>General Comments</u>

- Relaxation theories are useful to understand basic trends and to guide numerical experiments.

  Need improve relaxation ↔ dynamics connection

  → "routes to relaxation"

- There is a relation between turbulent relaxation and staircase formation, BUT:
- That relation is <u>not</u> simple and not fully understood → multi-stage relaxation scenario ?!
- Indeed, the precise meaning of "staircase" merits some care.

Re: “Staircases”

- Staircases are a well developed subject (prior 1972) and appear outside of GFD realm.
- Interesting and useful analytical models have been developed. More to the story than color VG’s
- There is a relationship between staircases and first order transition patterns.

<u>This Tutorial</u>

- Addresses both relaxation dynamics and staircase formation, aims to connect these
- Primarily <u>analytical</u> in approach → emphasis on variety of reduced models
- Primarily, though not exclusively, focused on applications to GFD, simple drift wave models
- Aims to relate/connect to MFE modelling issues
- Not a review

Collaborators

- Pei-Chun Hsu, Steve Tobias } minimum enstrophy relaxation
- Mischa Malkov, David Hughes } staircase models
- Ozgur Gurcan, Yusuke Kosuga, Zhibin Guo } jams theory
- Guilhem Dif-Pradalier: ExB staircase computations

Outline

- Basic Concepts – PV Dynamics and QG Flows
- Minimum Enstrophy Relaxation
  a) Model – Minimum Enstrophy Relaxation
  b) Rationale and Final States
  c) Dynamics → Structural Constraints on PV Flux
  d) Implications →

- Staircases (emphasis on formation)
  a) PV and otherwise
* b) Singular (Transport Bifurcation) Modulations
    a) Phillips, Balmforth and Beyond
    b) Return to QG
       → length scales, PV mixing, structures
    a) Jams and Jamming Waves
       → time delay

- Discussion
  -- just what is a staircase (apart pretty pics)?
  -- open theoretical issues?

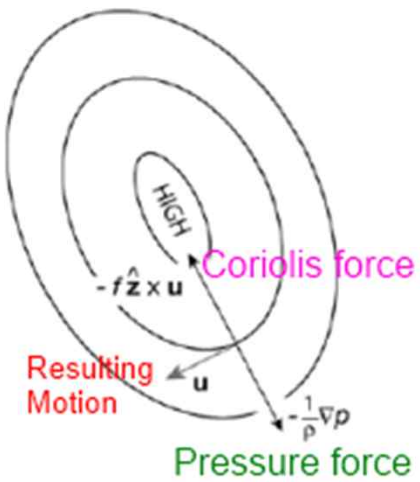
Basic Aspects of PV Dynamics

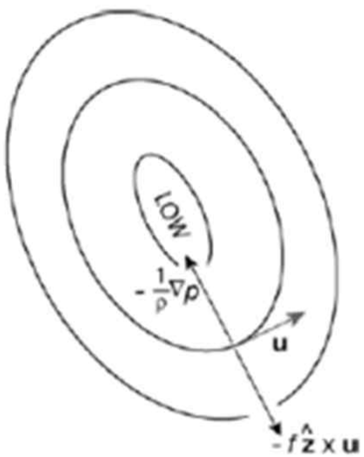
Geophysical fluids

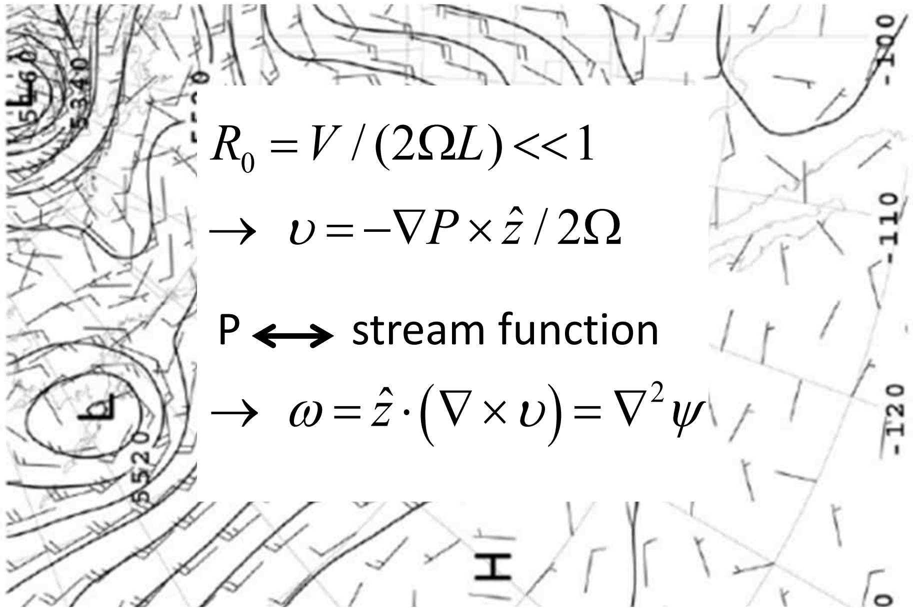
- Phenomena: weather, waves, large scale atmospheric and oceanic circulations, water circulation, jets…
- Geophysical fluid dynamics (GFD): low frequency ($\omega < \Omega$)

  "We might say that the atmosphere is a musical instrument on which one can play many tunes. High notes are sound waves, low notes are long inertial waves, and nature is a musician more of the Beethoven than the Chopin type. He much prefers the low notes and only occasionally plays arpeggios in the treble and then only with a light hand." – J.G. Charney

  ("Turing's Cathedral")
- Geostrophic motion: balance between the Coriolis force and pressure gradient







$$R_0 = V/(2\Omega L) << 1$$

$$\rightarrow \quad \upsilon = -\nabla P \times \hat{z} / 2\Omega$$

P ⟷ stream function

$$\rightarrow \quad \omega = \hat{z}\cdot(\nabla\times\upsilon) = \nabla^2\psi$$

Kelvin's theorem – unifying principle

- Kelvin's circulation theorem for rotating system

$$\oint v \cdot dl = \int (\nabla \times v + 2\Omega) \cdot \hat{z}\, dS \equiv C \qquad \dot{C} = 0$$

relative (for $\nabla \times v$), planetary (for 2Ω)

- Displacement on beta-plane

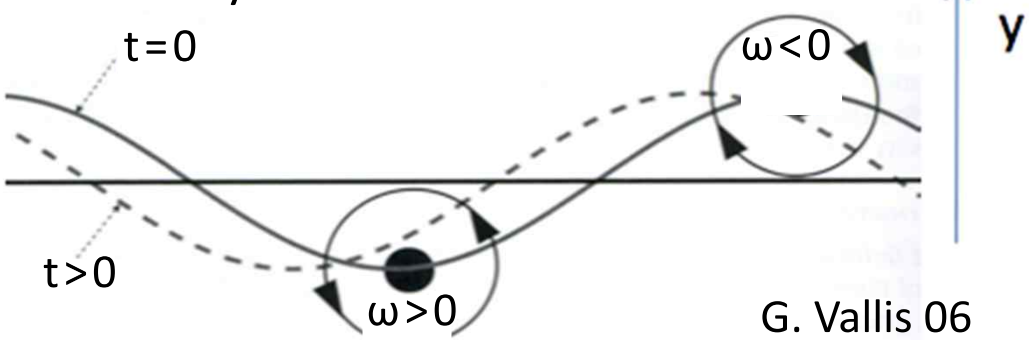
$$\dot{C} = 0 \;\rightarrow\; \frac{d}{dt}\nabla^2\psi = -2\Omega\cos\theta \frac{d\theta}{dt} = -\beta v_y$$

$$\beta = 2\Omega\cos\theta_0 / R_\oplus$$

- Quasi-geostrophic eq

$$\frac{d}{dt}\left(\nabla^2\psi + \beta y\right) = 0$$ **PV conservation**

→ Rossby wave



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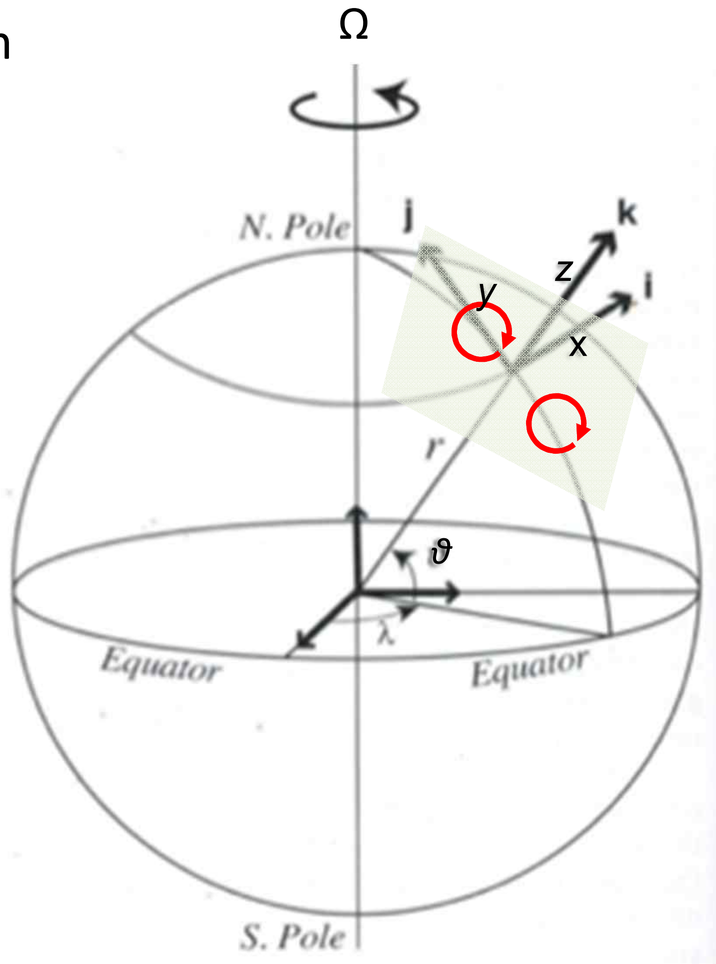


Drift wave model – Fundamental prototype

- Hasegawa-Wakatani : simplest model incorporating instability

$$V = \frac{c}{B}\hat{z}\times\nabla\phi + V_{pol}$$

$$J_\perp = n|e|V^i_{pol} \qquad \eta J_\parallel = -\nabla_\parallel\phi + \nabla_\parallel p_e$$

$$\nabla_\perp\cdot J_\perp + \nabla_\parallel J_\parallel = 0$$ → vorticity: $\rho_s^2\frac{d}{dt}\nabla^2\phi = -D_\parallel\nabla_\parallel^2(\phi - n) + \nu\nabla^2\nabla^2\phi$

$$\frac{dn_e}{dt} + \frac{\nabla_\parallel J_\parallel}{-n_0|e|} = 0$$ → density: $\frac{d}{dt}n = -D_\parallel\nabla_\parallel^2(\phi - n) + D_0\nabla^2 n$

→ in inviscid limit: **PV conservation** $\frac{d}{dt}\left(n - \nabla^2\phi\right) = 0$

→ PV flux = particle flux + vorticity flux

→ zonal flow being a counterpart of particle flux

QL: $\frac{\partial}{\partial t}\langle n\rangle = -\frac{\partial}{\partial r}\langle\tilde{v}_r\tilde{n}\rangle$

→ $\frac{\partial}{\partial t}\langle\nabla^2\phi\rangle = -\frac{\partial}{\partial r}\langle\tilde{v}_r\nabla^2\tilde{\phi}\rangle = -\frac{\partial^2}{\partial r^2}\langle\tilde{v}_r\tilde{v}_\theta\rangle$

- Hasegawa-Mima ($D_\parallel k_\parallel^2/\omega >> 1 \;\rightarrow\; n \sim \phi$)

$$\frac{d}{dt}\left(\phi - \rho_s^2\nabla^2\phi\right) + \upsilon_*\partial_y\phi = 0$$

PV conservation $\frac{dq}{dt}=0$

GFD: Quasi-geostrophic system	Plasma: Hasegawa-Wakatani system
$q=\nabla^2\psi+\beta y$ ↓ relative vorticity ($\nabla^2\psi$), ↓ planetary vorticity (βy)	$q=n-\nabla^2\phi$ ↓ density (guiding center) (n), ↓ ion vorticity (polarization) ($\nabla^2\phi$)
Physics: $\Delta y \rightarrow \Delta\left(\nabla^2\psi\right)$ →ZF	Physics: $\Delta r \rightarrow \Delta n \rightarrow \Delta\left(\nabla^2\phi\right)$ →ZF!

- Charney-Haswgawa-Mima equation

$$n=n_0+\tilde{n}$$

$$\tilde{n}\sim\frac{e\tilde{\phi}}{T}$$

H-W → H-M: $$\frac{1}{\omega_{ci}}\frac{\partial}{\partial t}\left(\nabla^2\phi-\rho_s^{-2}\phi\right)-\frac{1}{L_n}\frac{\partial}{\partial y}\phi+\frac{\rho_s}{L_n}J(\phi,\nabla^2\phi)=0$$

Q-G: $$\frac{\partial}{\partial t}\left(\nabla^2\psi-L_d^{-2}\psi\right)+\beta\frac{\partial}{\partial x}\psi+J(\psi,\nabla^2\psi)=0$$

PV Transport

- Zonal flows are generated by nonlinear interactions/mixing and transport.
- In x space, zonal flows are driven by Reynolds stress

$$\frac{\partial}{\partial t}\langle \upsilon_x \rangle = -\frac{\partial}{\partial y}\langle \tilde{\upsilon}_x \tilde{\upsilon}_y \rangle - \mu \langle \upsilon_x \rangle$$

Taylor's Identity

$$\langle \tilde{\upsilon}_y \tilde{q} \rangle = -\frac{\partial}{\partial y}\langle \tilde{\upsilon}_x \tilde{\upsilon}_y \rangle$$

→ **PV flux fundamental to zonal flow formation**

- Inhomogeneous PV mixing, not momentum mixing (dq/dt=0)

  → up-gradient momentum transport (negative-viscosity) not an enigma

- Reynolds stresses intimately linked to wave propagation

but: $\langle \tilde{\upsilon}_x \tilde{\upsilon}_y \rangle \rightarrow \sum_{\underline{k}} k_x k_y \left| \hat{\phi}_k \right|^2$

{ Wave-mixing, transport duality

$$\upsilon_{gy} = \frac{2 k_x k_y \beta}{(k^2)^2} \;,\quad S_y = \upsilon_{gy} \varepsilon$$

c.f. Review: O.D. Gurcan, P.D.; J. Phys. A (2015)
real space emphasis

Minimum Enstrophy Relaxation

Examples of Self-Organization Principles

$\rightarrow$ Turbulent Pipe Flow: (Prandtl $\rightarrow$ She)

$$\sigma = -\nu_T \frac{\partial \langle v_y \rangle}{\partial x} \qquad \nu_T \sim v_* x \qquad \Rightarrow \langle v_y \rangle \sim v_* \ln x$$

Streamwise Momentum undergoes mixing

The original "profile consistency"

$\rightarrow$ Magnetic Relaxation: (Woltjer-Taylor)

(RFP, etc) Minimize E_M at conserved global H_M $\Rightarrow$ Force-Free RFP profiles

* $\rightarrow$ PV Homogenization/Minimum Enstrophy: (Taylor, Prandtl, Batchelor, Bretherton, ...)

$\rightarrow$ PV tends to mix and homogenize

$\rightarrow$ Flow structures emergent from selective decay of potential enstrophy relative to kinetic energy

$\rightarrow$ Shakura-Sunyaev Accretion

$\rightarrow$ disk accretion enabled by outward viscous angular momentum flux

Observation

- Many commonalities - though NOT isomorphism - of magnetic and flow self-organization

- Specifically: Taylor Theory and Minimum Enstrophy Theory

	Magnetic (JB)	Flow (GI)
concept	topology	symmetry
process	turbulent reconnection	PV mixing
players	tearing modes, Alfven waves	drift wave turbulence
mean field	EMF = $\langle \tilde{v} \times \tilde{B} \rangle$	PV Flux = $\langle \tilde{v}_r \tilde{q} \rangle$
constraint	$\int d^3x \mathbf{A} \cdot \mathbf{B}$ conservation	Dual cascade (energy conservation)
outcome	B-profiles	(Zonal) flow

Foundation: Dual Cascade

- 2D turbulence conservation of energy and potential enstrophy

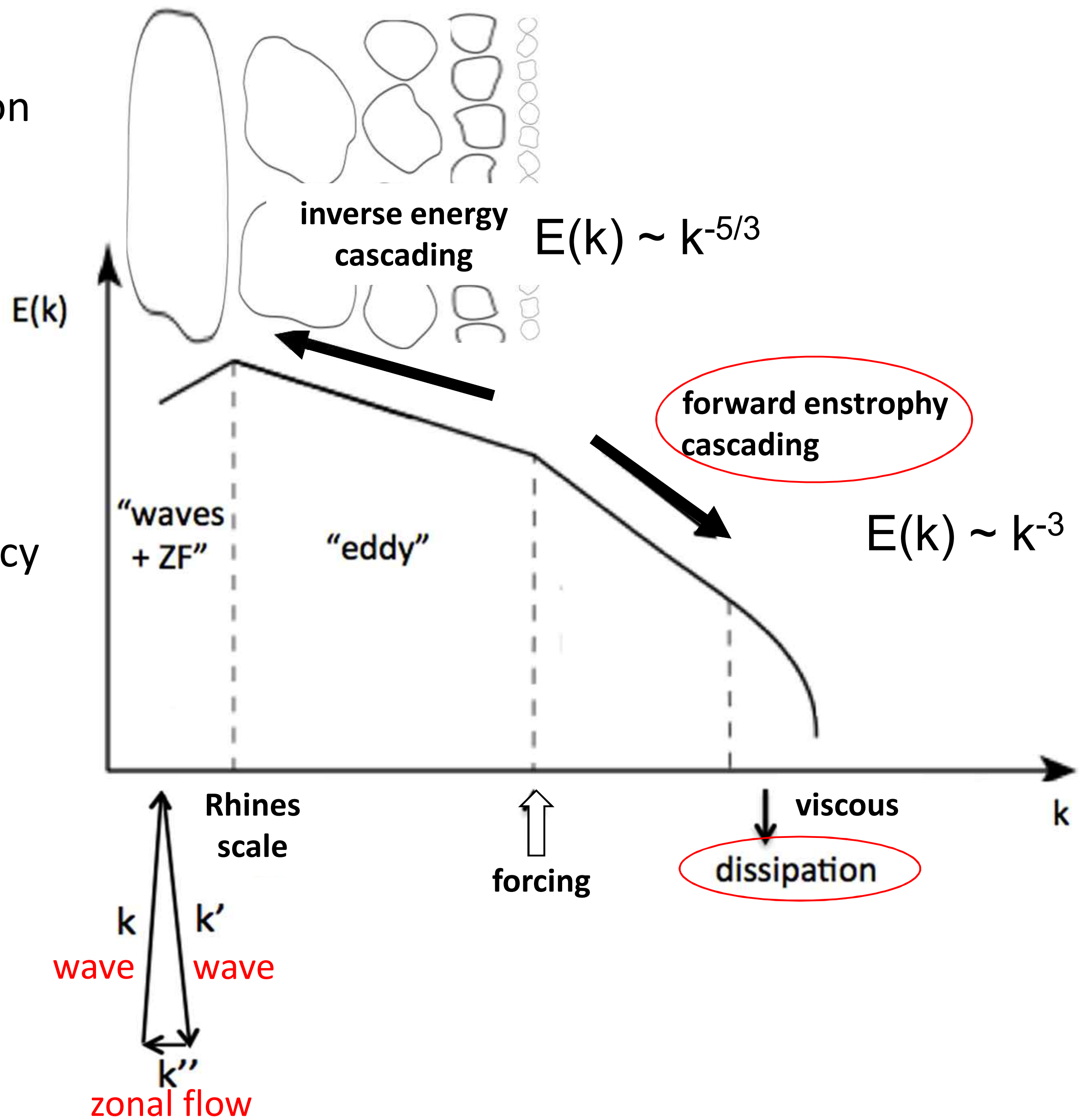
→ dual cascade (Kraichnan)

- When eddy turnover rate and Rossby wave frequency mismatch are comparable

$$\frac{\partial \omega}{\partial t} + \vec{u} \cdot \nabla \omega + \beta v = 0$$

$$\frac{U}{LT} \quad \frac{U^2}{L^2} \quad \beta U$$

→ Rhines scale $L_R \sim \sqrt{\frac{U}{\beta}}$



- Upshot : Minimum Enstrophy State

  (Bretherton and Haidvogel, 1976)

  -- idea : final state

  -- potential enstrophy forward cascades to viscous dissipation

  -- kinetic energy inverse cascades (drag?!)

-

-- calculate macrostate by minimizing potential enstrophy Ω subject to conservation of kinetic energy E, i.e.

$$\delta(\Omega + \mu E) = 0$$

[n.b. can include topography]

→ "Minimum Enstrophy Theory"

A Natural Question:

How exploit relaxation theory in dynamics?

Further Non-perturbative Approach for Flow!

– PV mixing in space is essential in ZF generation.

Taylor identity: $\left\langle \tilde{\upsilon}_y \nabla^2 \tilde{\phi} \right\rangle = -\partial_y \left\langle \tilde{\upsilon}_y \tilde{\upsilon}_x \right\rangle$

vorticity flux   Reynolds force

Key:
How represent inhomogeneous PV mixing

→

General structure of PV flux?
→relaxation principles!

most treatment of ZF:
-- perturbation theory
-- modulational instability
(test shear + gas of waves)
~ linear theory based

-> physics of evolved PV mixing?
-> something more general?

non-perturb model 1: use selective decay principle

What form must the PV flux have so as to dissipate enstrophy while conserving energy?

non-perturb model 2: use joint reflection symmetry

What form must the PV flux have so as to satisfy the joint reflection symmetry principle for PV transport/mixing?

Using selective decay for flux

	minimum enstrophy relaxation (Bretherton & Haidvogel 1976)	analogy ⟷	Taylor relaxation (J.B. Taylor, 1974)
turbulence	2D hydro		3D MHD
conserved quantity (constraint)	total kinetic energy		global magnetic helicity
dissipated quantity (minimized)	fluctuation potential enstrophy		magnetic energy
final state	minimum enstrophy state flow structure emergent		Taylor state force free B field configuration
structural approach	$\frac{\partial}{\partial t}\Omega < 0 \Rightarrow \Gamma_E \Rightarrow \Gamma_q$		$\frac{\partial}{\partial t}E_M < 0 \Rightarrow \Gamma_H$ (Boozer, '86)

dual cascade (conserved quantity / dissipated quantity)

- flux? what can be said about dynamics?

→ structural approach (this work): *What form must the PV flux have so as to dissipate enstrophy while conserving energy?*

General principle based on general physical ideas → useful for dynamical model

PV flux

→ PV conservation

mean field PV: $$\frac{\partial\langle q\rangle}{\partial t}+\partial_y\underbrace{\langle \upsilon_y q\rangle}=\nu_0\partial_y^2\langle q\rangle$$

Γ_q : mean field PV flux

Key Point:
form of PV flux Γ_q which dissipates enstrophy & conserves energy

selective decay

→ energy conserved $$E=\int\frac{(\partial_y\langle\phi\rangle)^2}{2}$$

$$\frac{\partial E}{\partial t}=\int\langle\phi\rangle\partial_y\Gamma_q=-\int\partial_y\langle\phi\rangle\Gamma_q \quad\Rightarrow\Gamma_q=\frac{\partial_y\Gamma_E}{\partial_y\langle\phi\rangle}$$

→ enstrophy minimized $$\Omega=\int\frac{\langle q\rangle^2}{2}$$

$$\frac{\partial\Omega}{\partial t}=-\int\langle q\rangle\partial_y\Gamma_q=-\int\partial_y\left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\phi\rangle}\right)\Gamma_E$$

$$\frac{\partial\Omega}{\partial t}<0\Rightarrow\Gamma_E=\mu\partial_y\left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\phi\rangle}\right)$$

μ ↓ parameter TBD; $\partial_y\langle\phi\rangle$ ↘ $\langle\upsilon_x\rangle$

$$\Rightarrow\Gamma_q=\frac{1}{\partial_y\langle\phi\rangle}\partial_y\left[\mu\partial_y\left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\phi\rangle}\right)\right]$$

general form of PV flux

Structure of PV flux

$$\Gamma_q = \frac{1}{\langle \upsilon_x \rangle}\partial_y\left[\mu\,\partial_y\left(\frac{\partial_y\langle q\rangle}{\langle \upsilon_x\rangle}\right)\right] = \frac{1}{\langle \upsilon_x \rangle}\partial_y\left[\mu\left(\frac{\langle q\rangle\partial_y\langle q\rangle}{\langle \upsilon_x\rangle^2}+\frac{\partial_y^2\langle q\rangle}{\langle \upsilon_x\rangle}\right)\right]$$

diffusion parameter calculated by perturbation theory, numerics…

drift and hyper diffusion of PV

<--> usual story : Fick's diffusion

relaxed state:

Homogenization of $\frac{\partial_y\langle q\rangle}{\langle \upsilon_x\rangle}$ → consistent with staircase

characteristic scale $\ell_c \equiv \sqrt{\left|\frac{\langle \upsilon_x\rangle}{\partial_y\langle q\rangle}\right|}$

$\ell > \ell_c$: zonal flow growth

$\ell < \ell_c$: zonal flow damping (hyper viscosity-dominated)

⟺

Rhines scale $L_R \sim \sqrt{\frac{U}{\beta}}$

$\ell > L_R$: wave-dominated

$\ell < L_R$: eddy-dominated

What sets the "minimum enstrophy"

- Decay drives relaxation. The relaxation rate can be derived by linear perturbation theory about the minimum enstrophy state

$$\langle q \rangle = q_m(y) + \delta q(y,t)$$
$$\langle \phi \rangle = \phi_m(y) + \delta\phi(y,t)$$
$$\partial_y q_m = \lambda \partial_y \phi_m$$
$$\delta q(y,t) = \delta q_0 \exp(-\gamma_{rel} t - i\omega t + iky)$$

γ_{rel} >0 relaxation

$$\gamma_{rel} = \mu\left(\frac{k^4 + 4\lambda k^2 + 3\lambda^2}{\langle \upsilon_x \rangle^2} - \frac{8q_m^2(k^2+\lambda)}{\langle \upsilon_x \rangle^4}\right)$$

$$\omega_k = \mu\left(-\frac{4q_m k^3 + 10 q_m k\lambda}{\langle \upsilon_x \rangle^3} - \frac{8q_m^3 k}{\langle \upsilon_x \rangle^5}\right)$$

- The condition of relaxation (modes are damped):

$$\gamma_{rel} > 0 \;\Rightarrow\; k^2 > \frac{8q_m^2}{\langle \upsilon_x \rangle^2} - 3\lambda \;\Rightarrow\; \boxed{\frac{8q_m^2}{\langle \upsilon_x \rangle^2} > 3\lambda}$$

→ Relates q_m^2 with ZF and scale factor

$$\Rightarrow \langle \upsilon_x \rangle^2 < \frac{3\lambda}{8q_m^2}$$

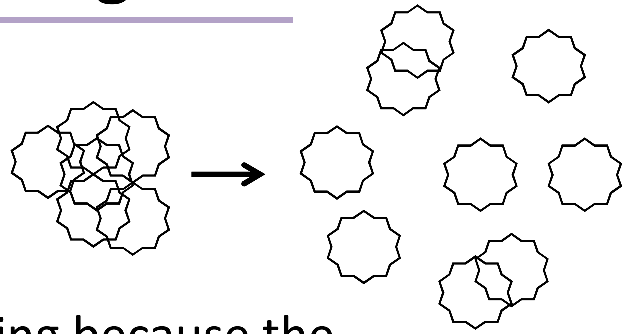
ZF can't grow arbitrarily large

$$\Rightarrow 8\, q_m^2 > \langle \upsilon_x \rangle^2\, 3\lambda$$

the 'minimum enstrophy' of relaxation, related to scale

Role of turbulence spreading

- Turbulence spreading: tendency of turbulence to self-scatter and entrain stable regime



- Turbulence spreading is closely related to PV mixing because the transport/mixing of turbulence intensity has influence on Reynolds stresses and so on flow dynamics.

- PV mixing is related to turbulence spreading

$$\frac{\partial E}{\partial t} = \int \langle \phi \rangle \partial_y \Gamma_q = -\int \partial_y \langle \phi \rangle \Gamma_q \qquad \Rightarrow \Gamma_q = \frac{\partial_y \Gamma_E}{\partial_y \langle \phi \rangle}$$

condition of energy conservation

- The effective spreading flux of turbulence kinetic energy

$$\Gamma_E = -\int \Gamma_q \langle v_x \rangle dy = -\int \frac{1}{\langle v_x \rangle} \partial_y \left[\mu \partial_y \left(\frac{\partial_y \langle q \rangle}{\langle v_x \rangle} \right) \right] \langle v_x \rangle dy = \mu \partial_y \left(\frac{\partial_y \langle q \rangle}{\langle v_x \rangle} \right) \Bigg|$$

→the gradient of $\partial_y \langle q \rangle / \langle v_x \rangle$, drives spreading

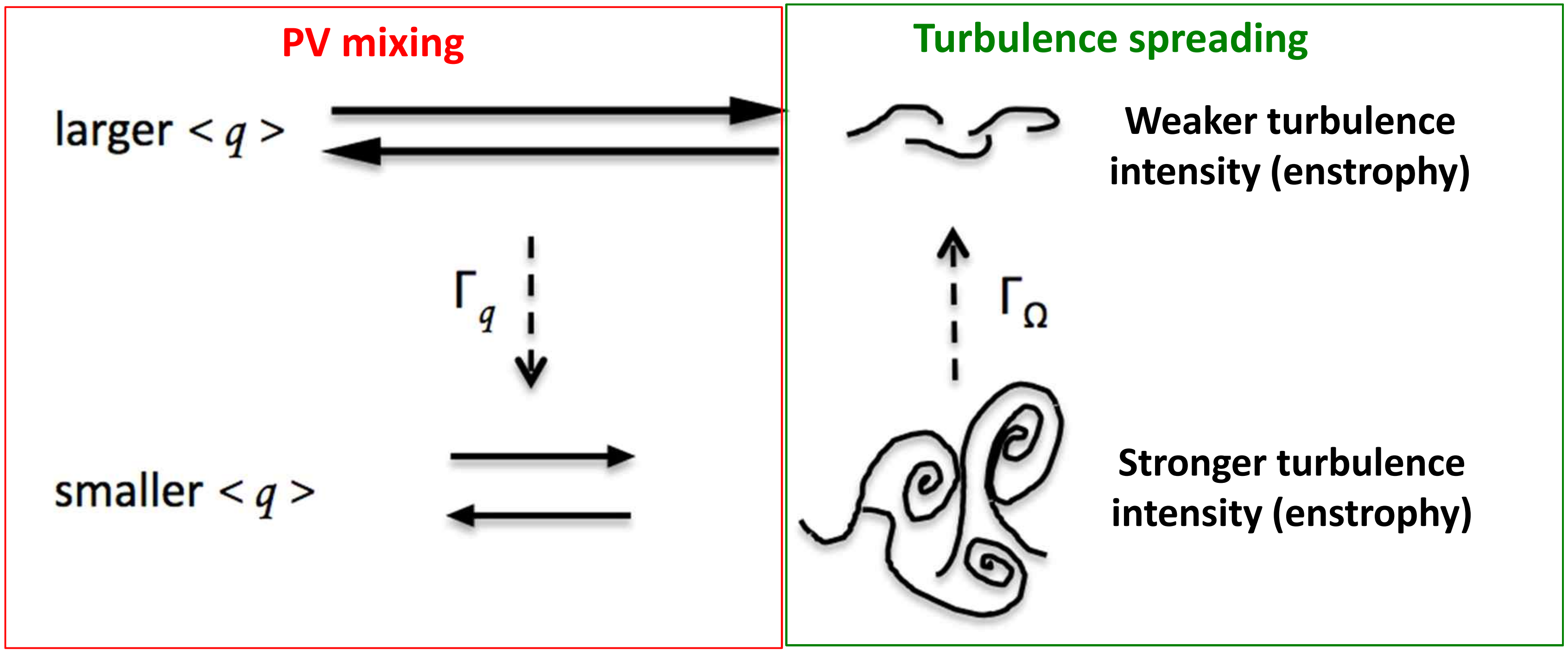
→ the spreading flux vanishes when $\partial_y \langle q \rangle / \langle v_x \rangle$ is homogenized

Discussion

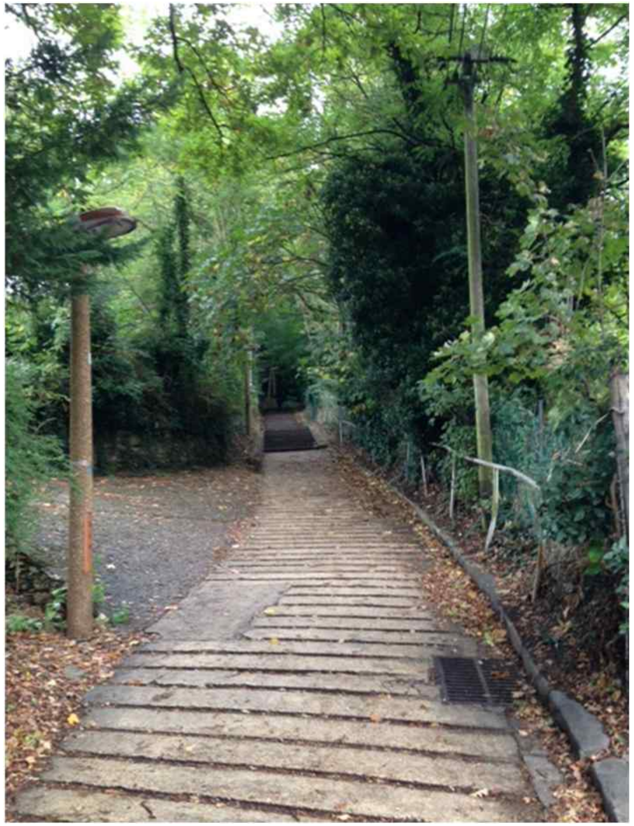
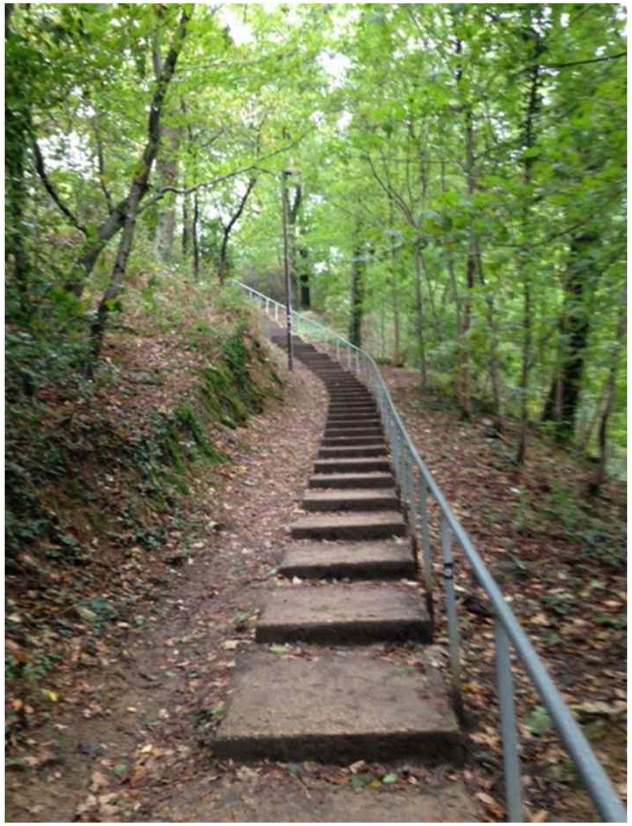
- PV mixing ↔ forward enstrophy cascade ↔ hyper-viscosity

  → How to reconcile effective negative viscosity with the picture of diffusive mixing of PV in real space?

- A possible explanation of up-gradient transport of PV due to turbulence spreading



Staircases

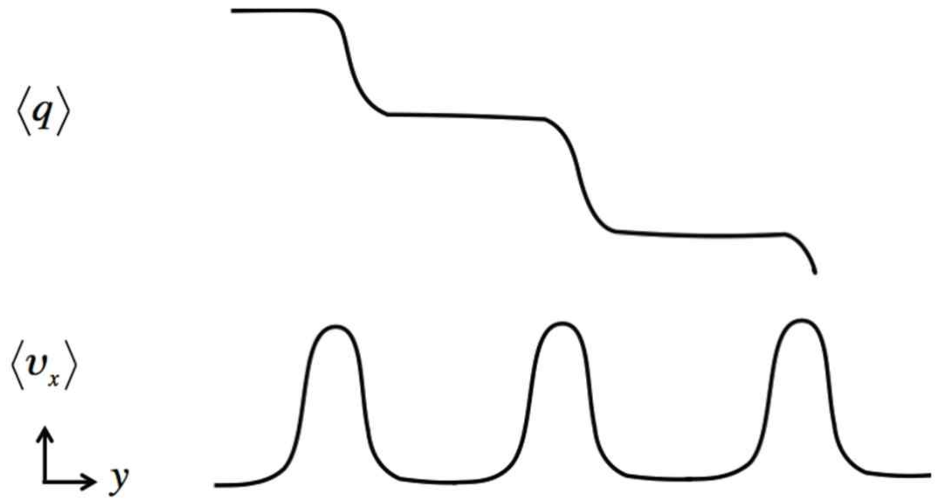



Staircases are prominent in French Academia

PV staircase

relaxed state: homogenization of $\frac{\partial_y \langle q \rangle}{\langle v_x \rangle}$ → PV gradient large where zonal flow large

→ Zonal flows track the PV gradient → PV staircase

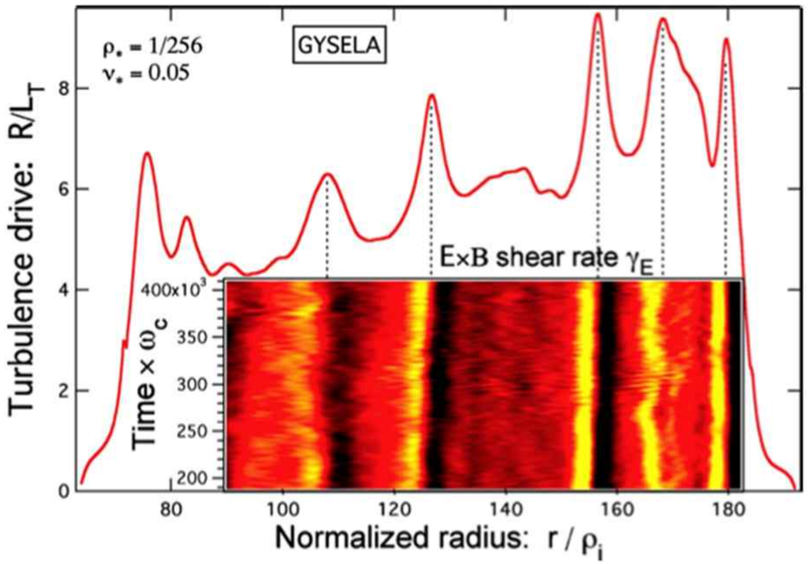


-- is a consistent solution

-- unusual degree of structure

- Highly structured profile of the staircase is reconciled with the homogenization or mixing process required to produce it.
- Staircase may arise naturally as a consequence of minimum enstrophy relaxation.

Some Required Pictures of Self-Organization in France



Legion imitating a zonal flow

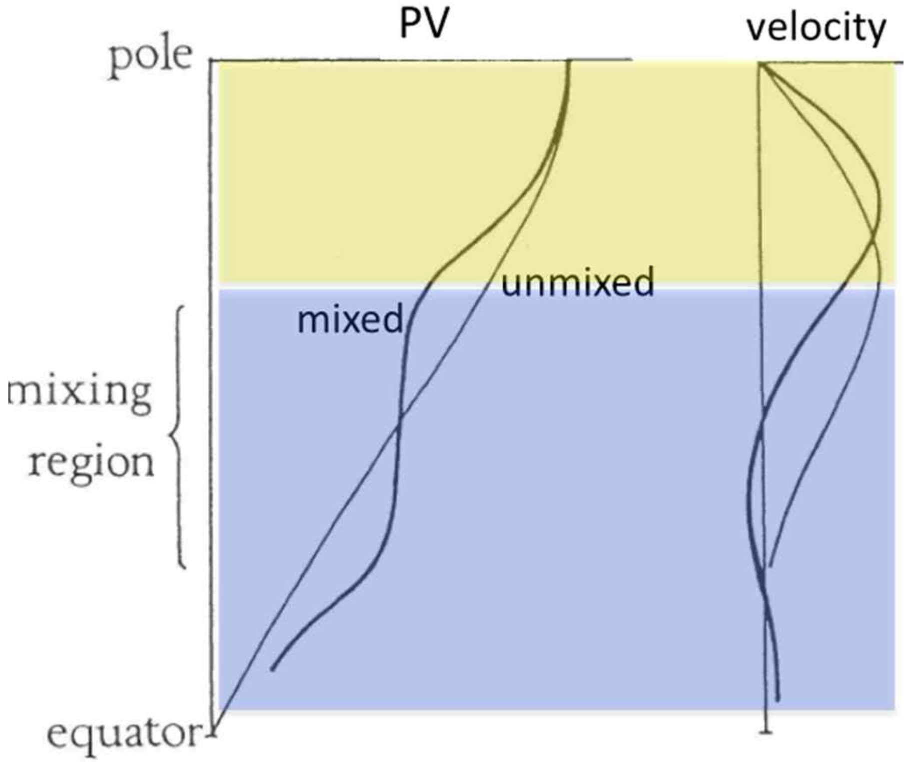
- E x B staircase (GDP, PD et al. 2010)
- driven system
- quasi-periodic E x B shear layers and $\nabla T_i / T_i$ corrugations
- step-scale → avalanche outer scale
- Not correlated with q

n.b. 2010 paper written under UCSD by-line …

How make a step? → Inhomogeneous PV mixing

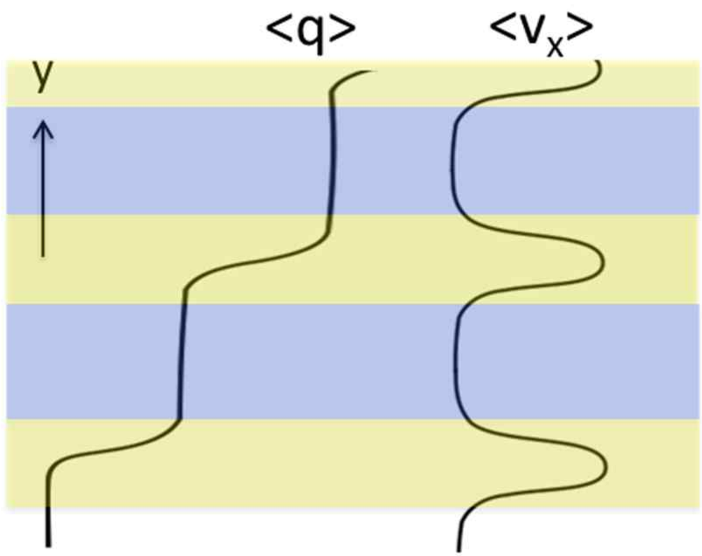
- **PV mixing is the fundamental mechanism for zonal flow formation**

$$\delta(\mathrm{PV}) \rightarrow \delta(\nabla^2\psi) \rightarrow \delta(\psi) \rightarrow \upsilon = \nabla \times \psi$$



McIntyre 1982
Dritschel & McIntyre 2008

→ **PV staircacse**



N.B. for ITG turbulence:

$$PV \rightarrow \alpha\ \hat{T}/T + \nabla^2\phi$$

$$\delta PV = 0 \rightarrow\ \delta\left(\hat{T}/T\right) \Rightarrow -\delta\left(\nabla^2\phi\right)$$

and shear flow formation

"What is the difference between a staircase and a nonlinear wave, and why would anyone care?"

- Senior UCSD Experimentalist

This is a staircase:

i.e. clear scale ordering

$w < \Delta_{step} \ll L_{sys}$ → barrier

↑

step layer width

→ developed modulational wave

N.B.:

- Also mechanism:
  - Staircase → first order
  - Wave → second order
- Beware: both tilt eddys, shear, etc.

This is a nonlinear wave:

i.e. $w \sim \Delta$

c.f. Fujisawa, et al, mid 90’s et. seq.

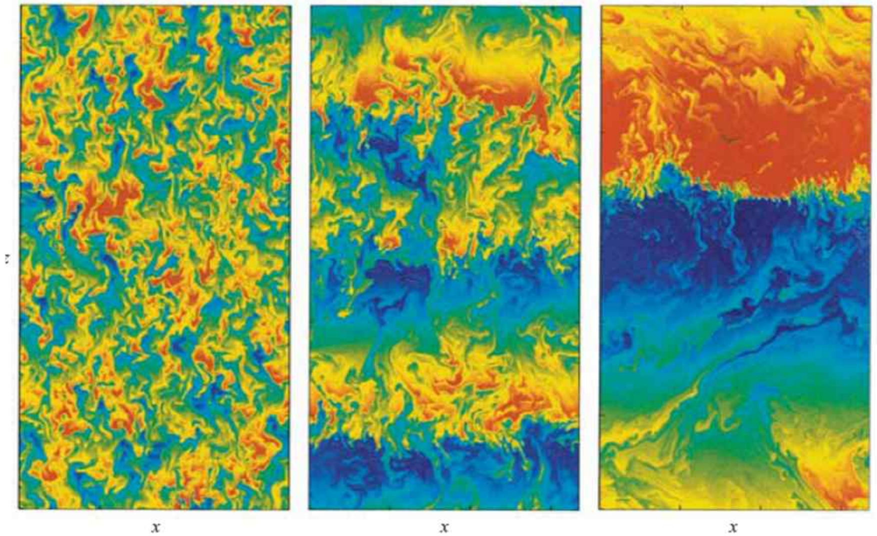
- ➢ Staircases are much more ubiquitous than in GK turbulence

- – Stably stratified turbulence (late ‘60s) (ocean surface layer)
- – Thermohaline convection
- – Driven QG
- – MHD (magneto-convection, magnetic buoyancy)

- – All involve formation of sharp gradient steps, by mixing processes.
- – Not all involve “shear suppression”, etc.

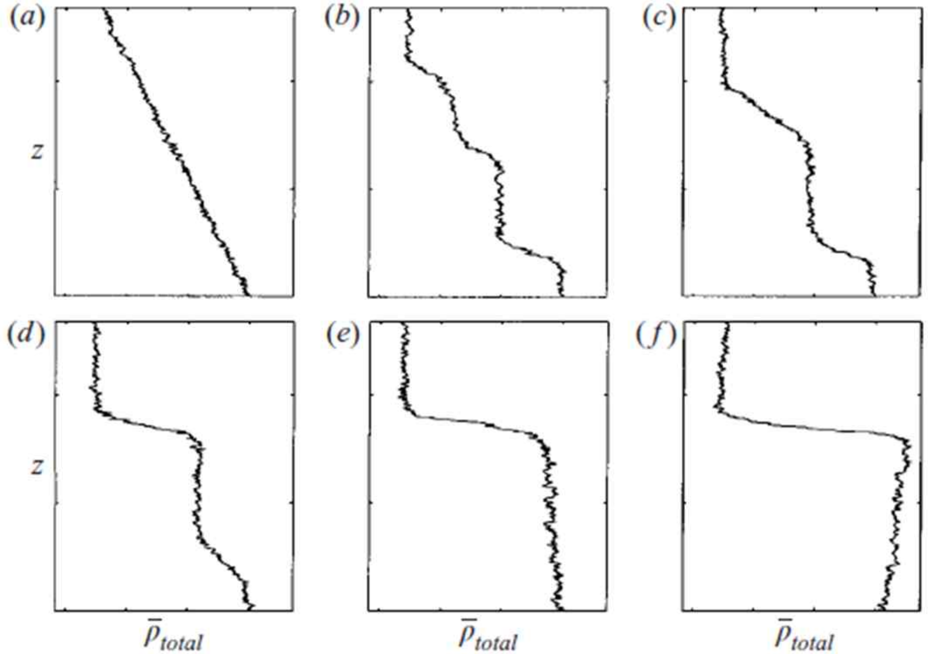
→ General phenomena

Thermohaline Layer Simulation (Radko, 2003)



Sharp interface formed
colors → salt concentration

Staircase



Single layer

Staircase formed, followed
by 'condensation' to single layer
→ Merger events

- What is a staircase?
- Cf Phillips'72:

(other approaches possible)

SHORTER CONTRIBUTION

Turbulence in a strongly stratified fluid—is it unstable?

O. M. PHILLIPS*

(*Received* 30 *July* 1971; *in revised form* 6 *October* 1971; *accepted* 6 *October* 1971)

Abstract—It is shown that if the buoyancy flux is a local property of turbulence in a stratified fluid that decreases sufficiently rapidly as the local Richardson number increases, then an initially linear density profile in a turbulent flow far from boundaries may become unstable with respect to small variations in the vertical density gradient. An initially linear profile will then become ragged; this possible instability may be associated on occasions with the formation of density microstructure in the ocean.

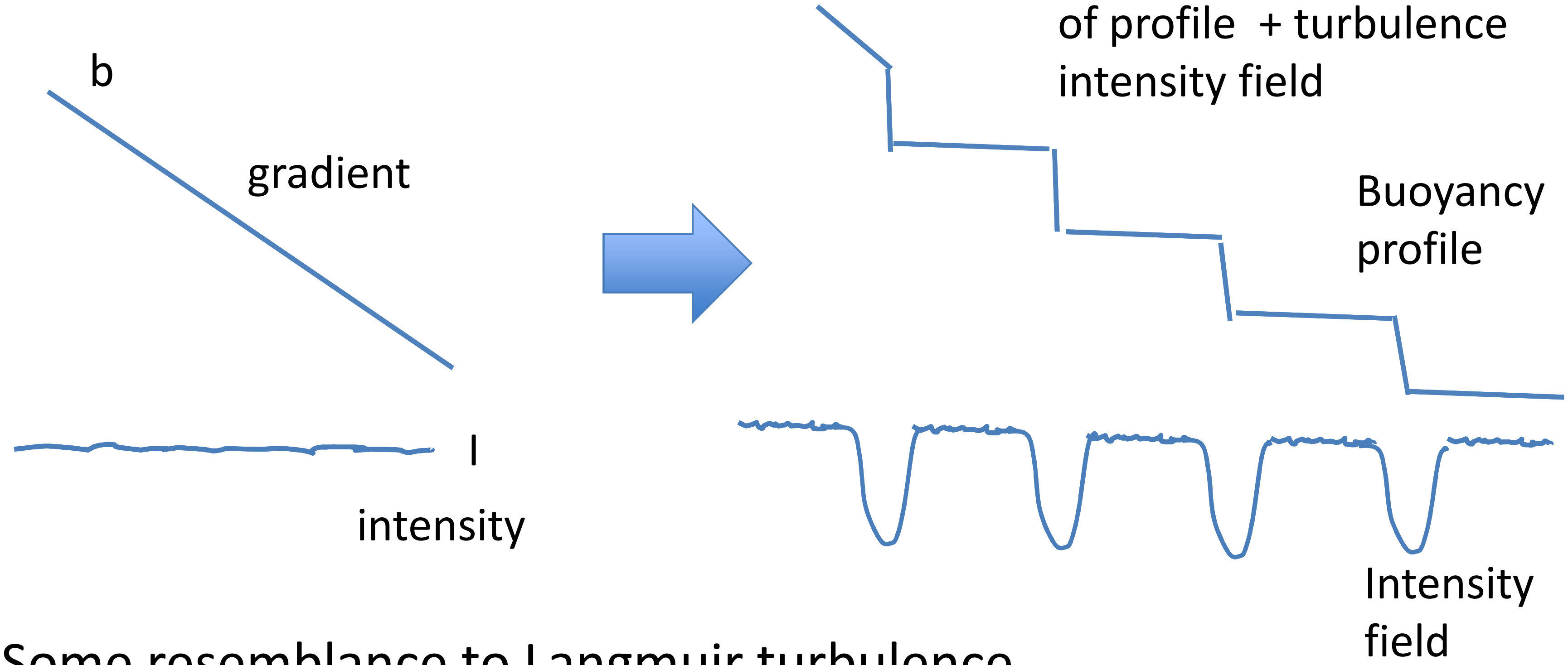
- Instability of mean + turbulence field requiring:

  $\delta\Gamma_b/\delta Ri < 0$; flux dropping with increased gradient

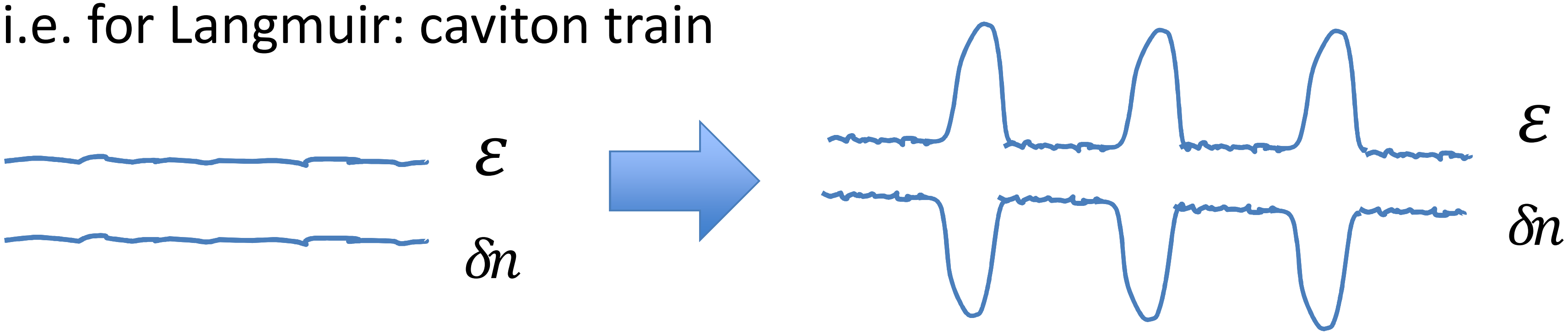
  $\Gamma_b = -D_b \nabla b$, $Ri = g\nabla b/(v')^2$

- Obvious similarity to transport bifurcation

In other words:



Some resemblance to Langmuir turbulence
i.e. for Langmuir: caviton train



$$\delta n / n \approx -\varepsilon$$

- OK: Is there a “simple model” encapsulating the ideas?
- Balmforth, Llewellyn-Smith, Young 1998 → staircase in stirred stably stratified turbulence
- Idea: 1D $K - \epsilon$ model
  - turbulence energy; with production, dissipation spreading

+

  - Mean field evolution
  - Diffusion: $\tilde{V}\, l_m \sim (\epsilon)^{\frac{1}{2}}\, l_{mix}$
  - l_{mix} → mixing length ?!

- What is l_{mix} ?

$$1/\,l^2 = 1/\,l_f^2 + 1/\,l_{oz}^2$$

l_{oz} : ~ Ozmidov scale

~ balance of buoyancy production vs. dissipation

i.e. $\tilde{V}^3/l \sim g\langle\tilde{V}\ \delta b\rangle$

$$\delta b \sim \left(\tilde{V}/(\tilde{V}/l)\right)\partial b/\partial z$$

➔ $1/l_{oz} \approx (b_z/e)^{1/2}$

<u>or</u> $V(l)/l \sim N$ ➔ l_{oz}

➔ smallest "stratified" scale

{ System mixes at steady state on scale of energy balance

N.B.: $b_z \uparrow$, $e \downarrow$ → $l \downarrow$

$e \approx \langle\tilde{V}^2\rangle$ energy

The model

- Mean Field:

$$D = e^{1/2} l$$

$$1/l^2 = 1/\, l_f^2 + 1/l_{oz}^2$$

$$\partial_t b = \partial_z (D \partial_z b)$$

$$e = \langle \tilde{V}^2 \rangle$$

N.B.: Not a typo! No residual molecular diffusion!

- Fluctuations:

spreading

Production $g\langle \tilde{V} \delta\rho \rangle$

forcing $F \sim \sqrt{e}\,(u_0^2 - e)$

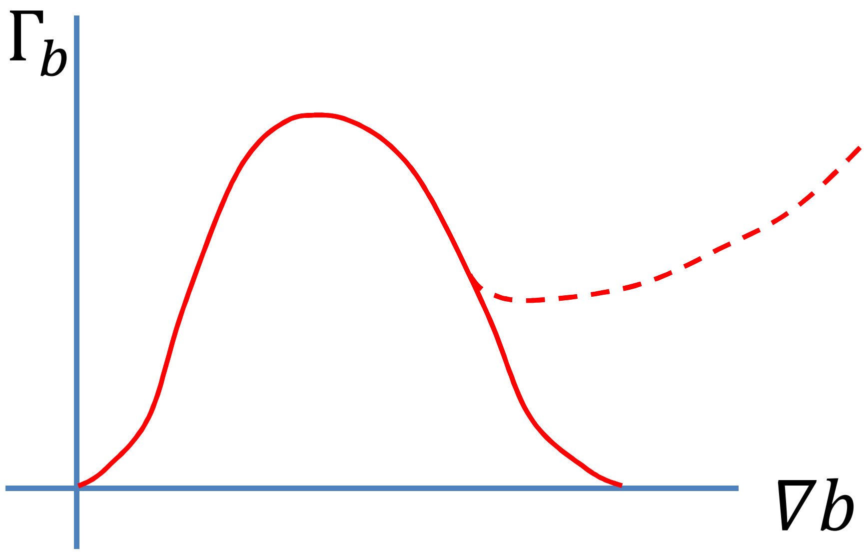
$$\partial_t e = \partial_z D \partial_z e \; - l e^{\frac{1}{2}} \partial_z b \; - \frac{e^{\frac{3}{2}}}{l} + F$$

dissipation

N.B. $\partial_t \left(\int [e - zb\,] \right) = 0$ (energy balance)

- <u>Some observations</u>
  - No molecular diffusion branch ("neoclassical H-mode")
    Steep b_z balanced by dissipation, reduced l
  - Step layer set by turbulence spreading (N.B. interesting lesson for case when D_{neo} feeble – i.e. particles)
  - Forcing acts to initiate fluctuations, but production ($\sim b_z$) is the main driver
  - Gradient-fluctuation energy balance is crucial
  - Can explore stability of initial uniform e, b_z field → akin modulation problem

- The physics: Negative Diffusion



"H-mode" like branch
(i.e. residual collisional diffusion)
is not input

- Usually no residual diffusion
- 'branch' upswing → nonlinear processes

- Instability driven by local transport bifurcation
- $\delta\Gamma_b/\delta\nabla b < 0$

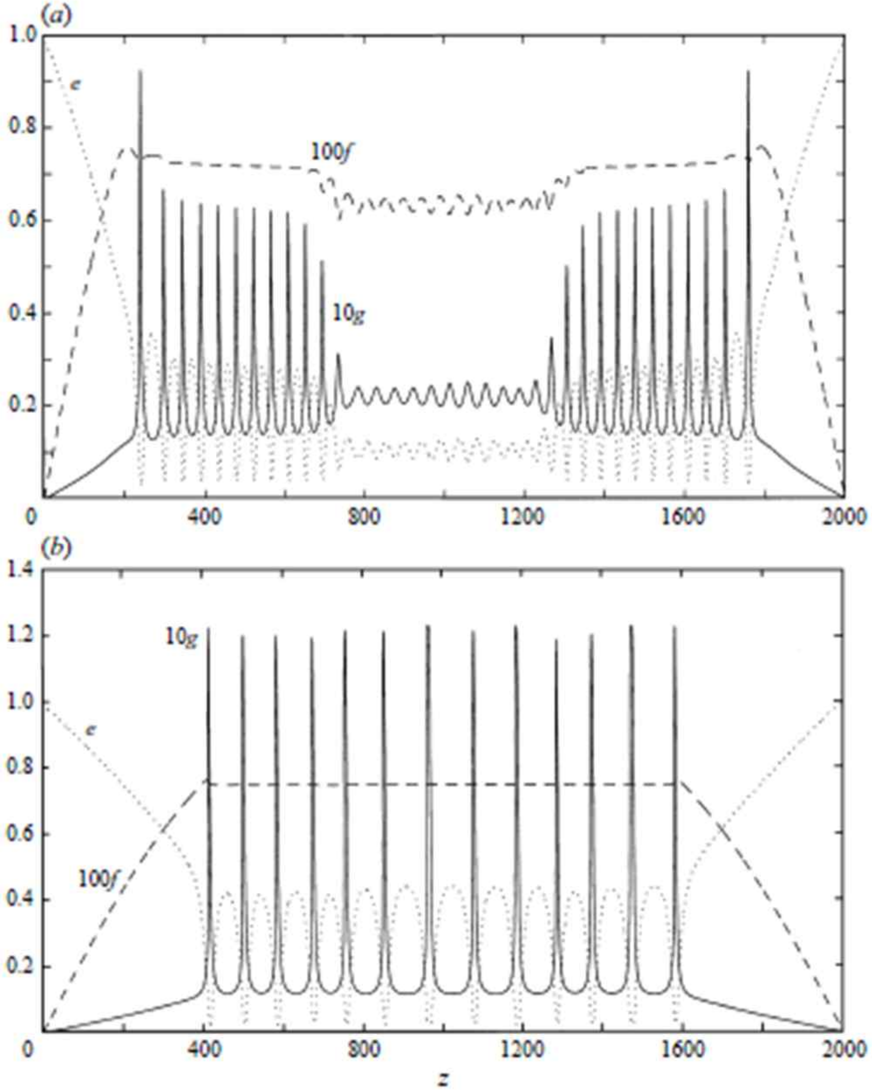
  ➔ 'negative diffusion'

Negative slope
Unstable branch

- Feedback loop $\Gamma_b \downarrow \rightarrow \nabla b \uparrow \rightarrow I \downarrow \rightarrow \Gamma_b \downarrow$

Critical element:
$l \rightarrow$ mixing length

- <u>Some Results</u>

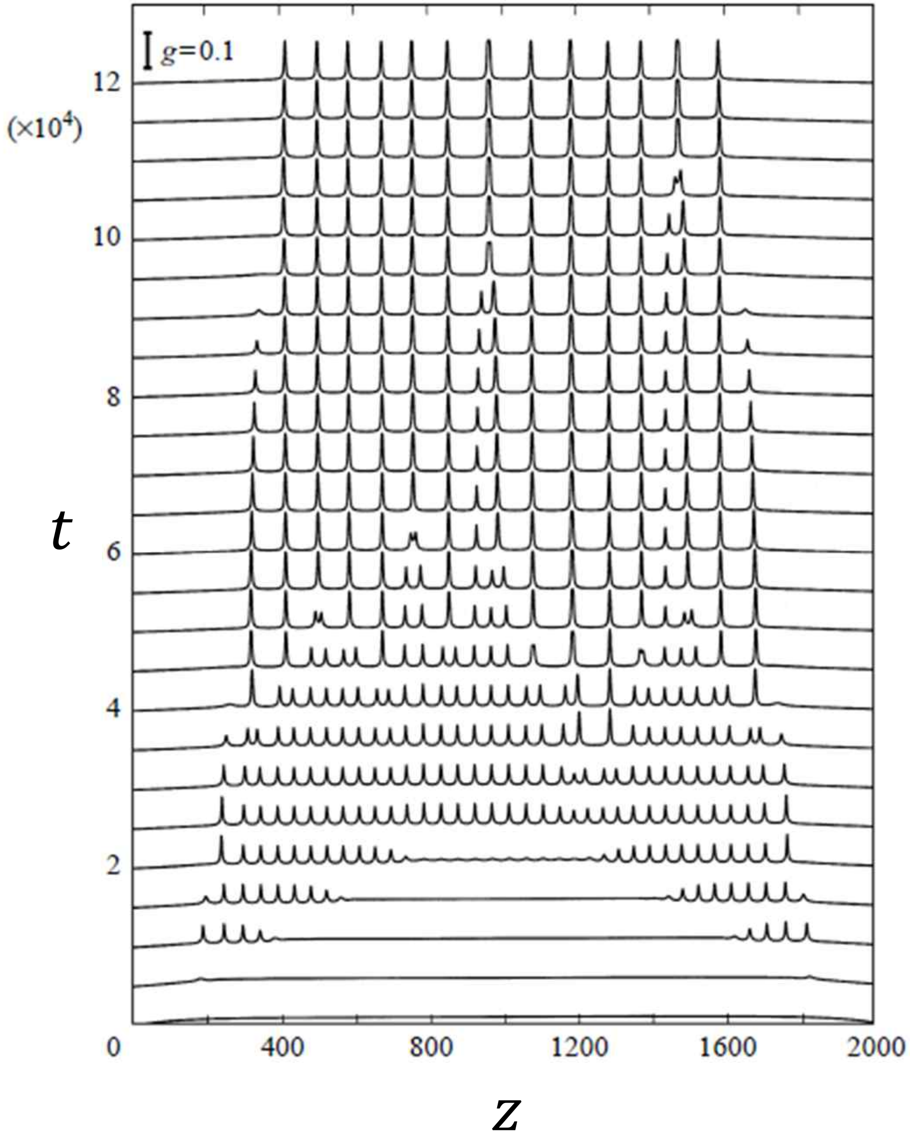


Plot of b_z (solid) and e (dotted) at early time. Buoyancy flux is dashed → near constant in core

Later time → more akin expected "staircase pattern". Some <u>condensation</u> into larger scale structures has occurred.

- <u>Time Evolution</u>

b_z $(\times 10^4)$



- Time progression shows merger process – akin bubble competition for steps
- Suggests trend to merger into fewer, larger steps
- Relaxation description in terms of merger process!? i.e. population evolution
- Predict/control position of final large step?

To QG

- PV staircases observed in nature, and in the unnatural (i.e. codes)
- Formulate 'minimal' dynamical model ?! (n.b. Dritschel-McIntyre 2008 does not address dynamics)

Observe:

- 1D adequate: for ZF need 'inhomogeneous PV mixing' + 1 direction of symmetry
- Best formulate intensity dynamics in terms potential enstrophy $\epsilon = \langle \tilde{q}^2 \rangle$
- Length? : $\Gamma_q \partial \langle q \rangle / \partial y \sim \tilde{q}^3$ (production-dissipation balance)
- $\rightarrow l \sim \langle \tilde{q}^2 \rangle^{\frac{1}{2}} / \partial \langle q \rangle / \partial y \sim l_{Rhines}$

Model

$$\partial_t \langle q \rangle = \partial_y D \partial_y \langle q \rangle$$

Dissipation

Mean Field

$$\partial_t \epsilon - \partial_y D \partial_y \epsilon = D\left(\partial_y \langle q \rangle\right)^2 - \epsilon^{\frac{3}{2}} + F$$

Forcing

Spreading

Production

Where:

Fluctuations

$$\frac{1}{l^2} = \frac{1}{l_f^2} + \frac{1}{l_{Rh}^2}$$

$$l_{Rh}^2 = \epsilon / \left(\partial_y \langle q \rangle\right)^2$$

$$D \sim l^2 \sqrt{\epsilon}$$

$\partial_t \left(\frac{\langle q \rangle^2}{2} + \epsilon \right) = 0$, to forcing, dissipation

Aside

- What of wave momentum?

- PV mixing $\leftarrow\rightarrow D\partial_y\langle q\rangle$

  So $\rightarrow \langle\tilde{V}\tilde{q}\rangle \rightarrow \partial_y\langle\tilde{V}_y\tilde{V}_x\rangle \rightarrow$ R.S.

- But:

  R.S. $\leftarrow\rightarrow \langle k_x k_y\rangle \leftarrow\rightarrow V_{gy}E$

➔ Feedback:

$$\langle q\rangle' \uparrow \rightarrow\ l \downarrow \rightarrow \epsilon \downarrow \rightarrow D \downarrow$$

(Production)

Alternative

- Note: $l^2 = \frac{1}{1+1/l_{Rh}^2} \rightarrow \frac{1}{1+\langle q \rangle'^2 / \epsilon}$ $(l_f \sim 1)$
- Reminiscent of weak turbulence perspective:

$$D = D_{pv} = \sum_{\vec{k}} \frac{\langle \tilde{V}^2 \rangle \Delta\omega_{\vec{k}}}{\omega_{\vec{k}}^2 + \Delta\omega_{\vec{k}}^2}$$

$$\omega_{\vec{k}} = -k_x \langle q \rangle' / k^2$$

$$\Delta\omega_{\vec{k}} \approx k\tilde{V}_{\vec{k}}$$

Ala' Dupree'67:

$$D_{pv} \approx \frac{1}{k^2} \left(\sum_{\vec{k}} k^2 \langle \tilde{V}^2 \rangle_{\vec{k}} - \frac{k_x^2 \left(\langle q \rangle' \right)^2}{(k^2)^2} \right)^{1/2}$$

Steeper $\langle q \rangle'$ quenches diffusion

$$D_{pv} \approx \frac{l_0^2\ \epsilon^{\frac{1}{2}}}{1 + \frac{l_0^2}{\epsilon}(\langle q \rangle')^2}$$

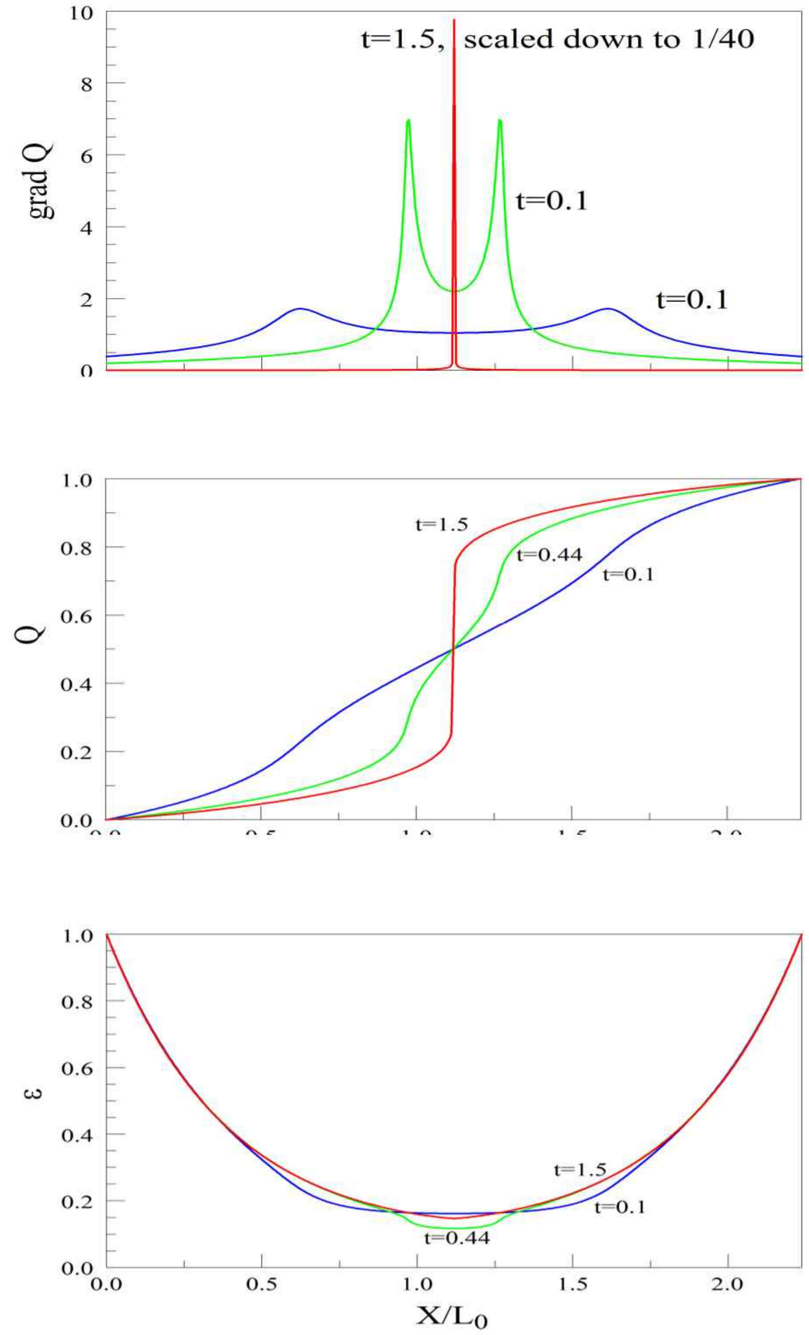
- ω vs $\Delta\omega$ dependence gives D_{pv} roll-over with steepening
- Rhines scale appears naturally
- Recovers effectively same model

Physics:

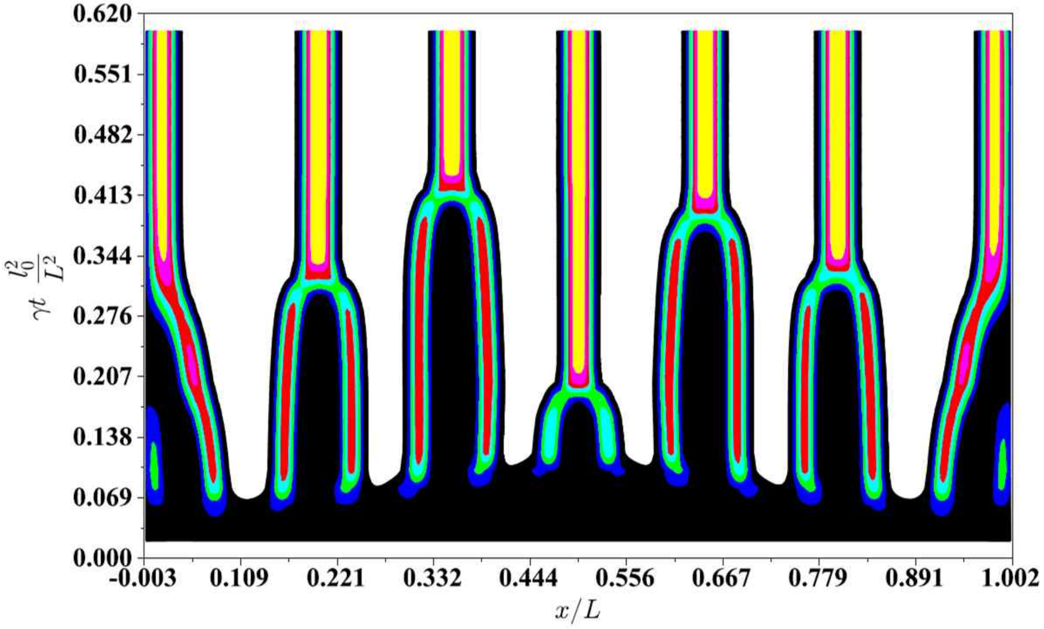
① "Rossby wave elasticity' (MM) → steeper $\langle q \rangle'$ → stronger memory

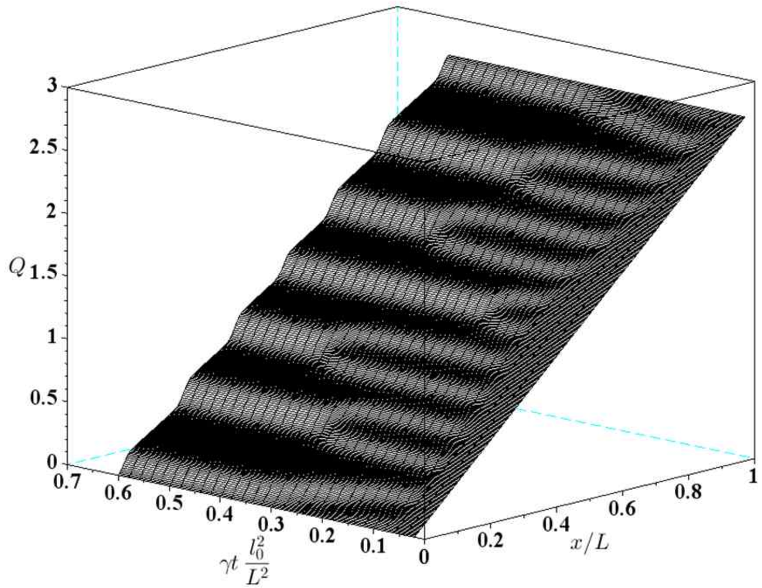
② Distinct from shear suppression

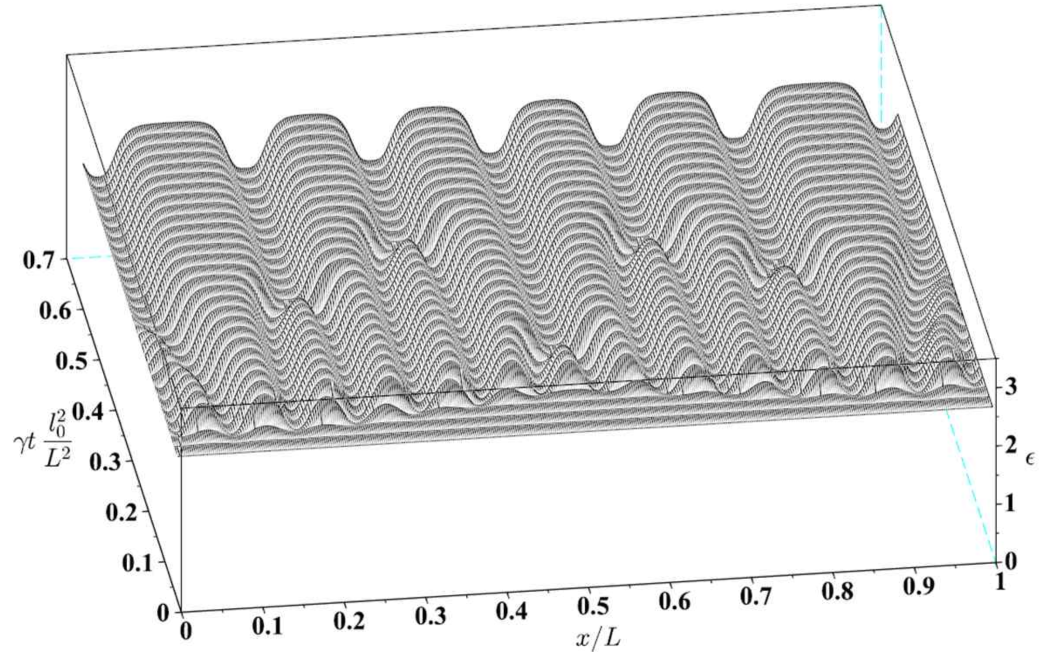
Numerical Results



- collapse of two steps into one
- appears to proceed to infinity
- eps_0=0.1

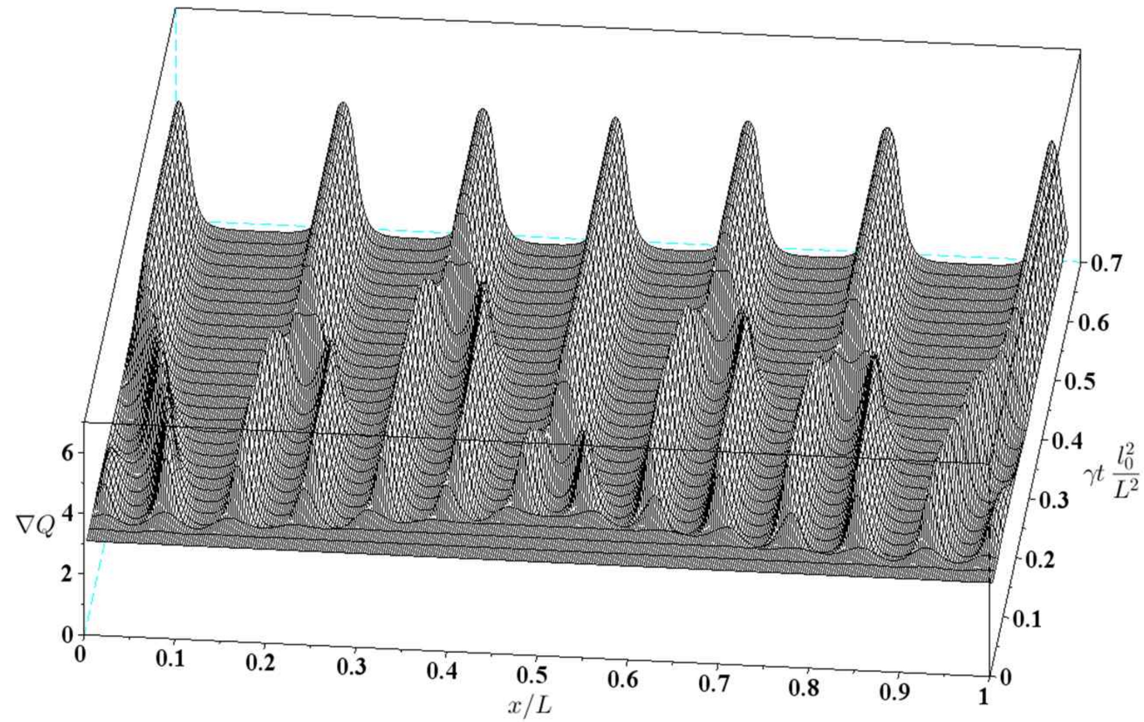




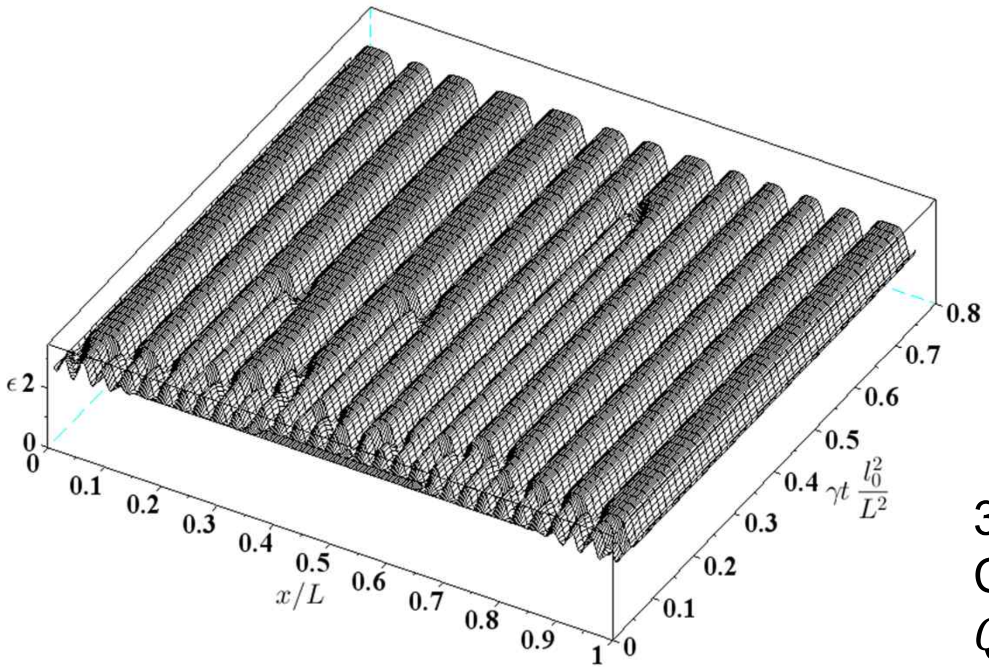


14 → 7 coalescence
Contour plot shows grad Q
Q, ϵ fixed at boundaries

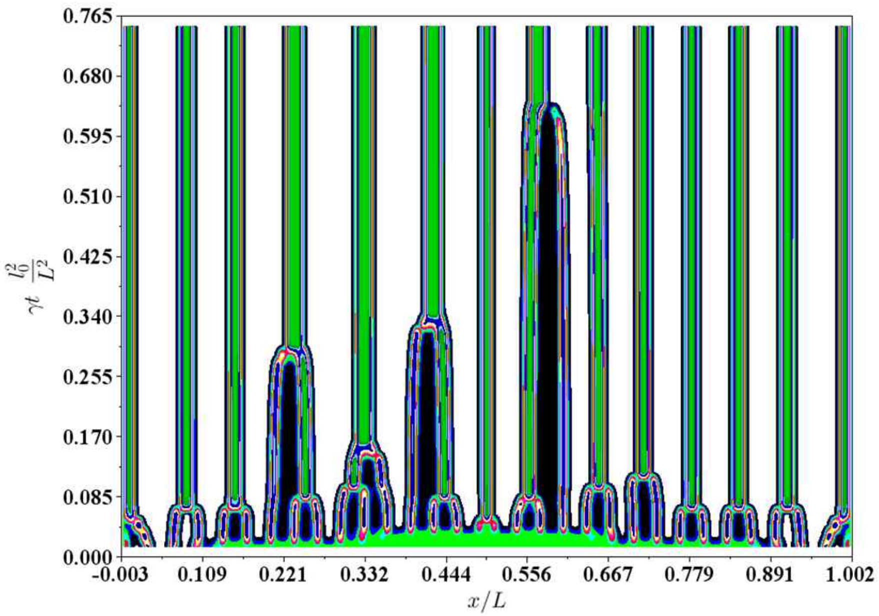
Parameters:
L^2=10^5
kappa =3



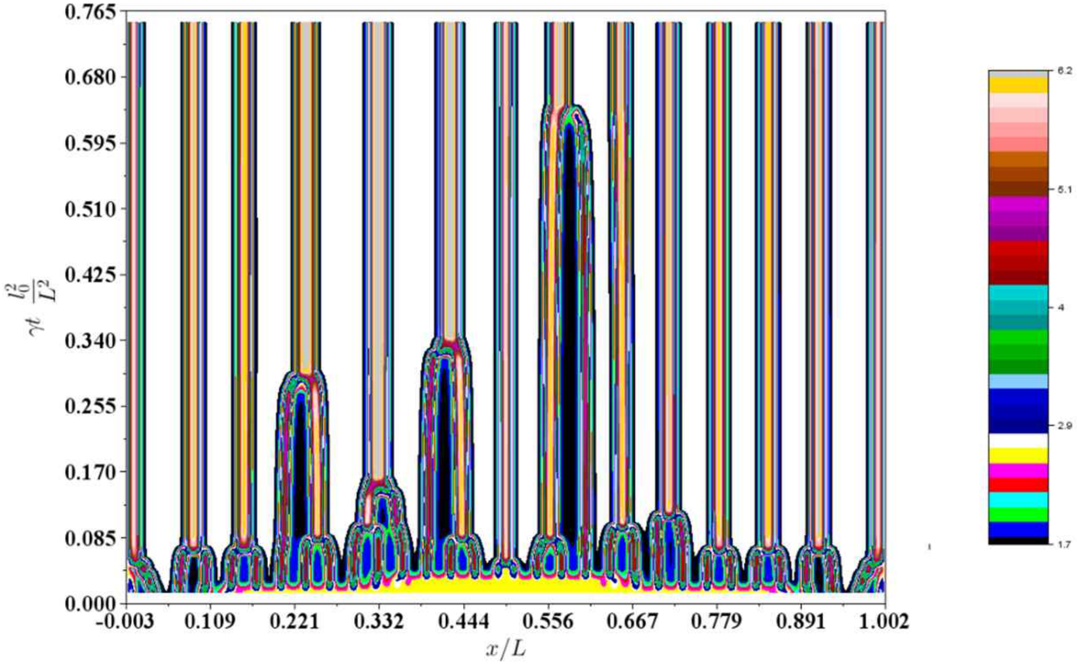
gradQ shown as a
contour plot in preceding
VG

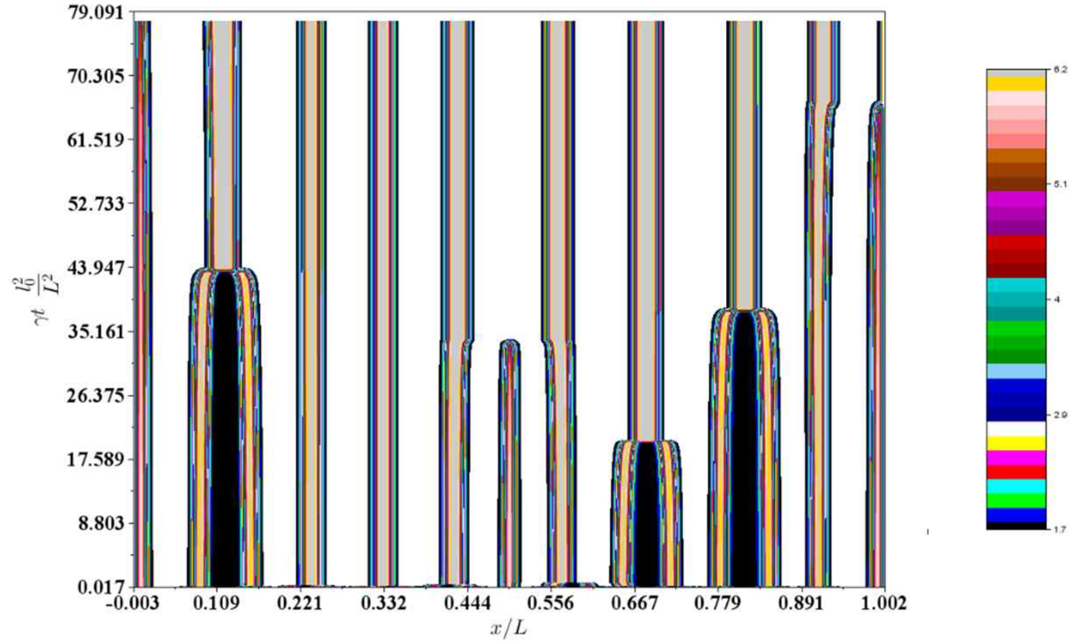


30 --> 14 coalescence
Contour plot shows grad Q
Q, ϵ fixed at boundaries



Parameters:
L^2=5.45*10^5
kappa =3 (unstable equilibrium)
eps_0=2.23


0.765
0.680
0.595
0.510
0.425
0.340
0.255
0.170
0.085
0.000
$\gamma t \frac{l_0^2}{L^2}$
-0.003
0.109
0.221
0.332
0.444
0.556
0.667
0.779
0.891
1.002
x/L
6.2
5.1
4
2.9
1.7


79.091
70.305
61.519
52.733
43.947
35.161
26.375
17.589
8.803
0.017
$\gamma t \frac{l_0^2}{L^2}$
-0.003
0.109
0.221
0.332
0.444
0.556
0.667
0.779
0.891
1.002
x/L
6.2
5.1
4
2.9
1.7

What of Regimes with Avalanching?

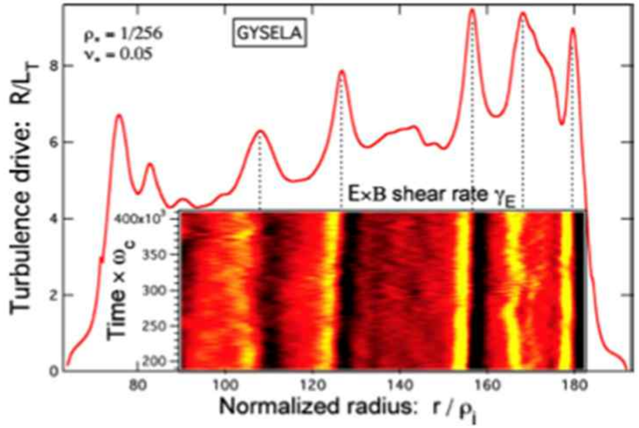
→ Jams and Jamitons

Highlights

Observation of ExB staircases

→ Failure of conventional theory of avalanches

(emergence of particular scale???)



Model extension from Burgers to telegraph

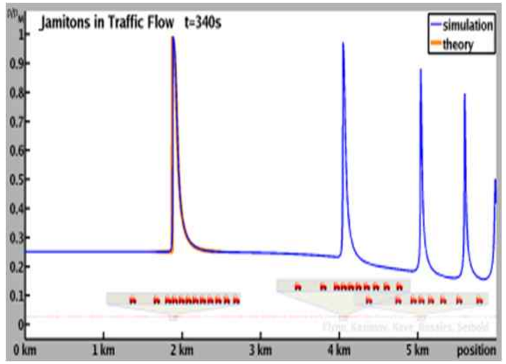
$$\partial_t \delta T + \lambda \delta T \partial_x \delta T = \chi_2 \partial_x^2 \delta T$$

$$\Rightarrow \tau \partial_t^2 \delta T + \partial_t \delta T + \lambda \delta T \partial_x \delta T = \chi_2 \partial_x^2 \delta T$$

finite response time → like drivers' response time in traffic

Analysis of telegraph eqn. predicts heat flux jam

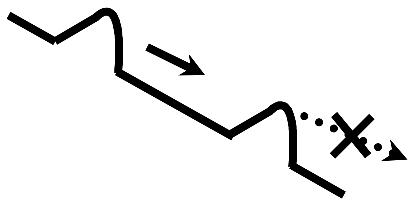
- scale of jam comparable to staircase step



Towards a model

- How do we understand quasi-regular pattern of ExB staircase, generated from stochastic heat avalanche???

- An idea: jam of heat avalanche



corrugated profile ↔ ExB staircase

→ accumulation of heat increment
→ stationary corrugated profile

→ corrugation of profile occurs by 'jam' of heat avalanche flux

* → time delay between $Q[\delta T]$ and δT is crucial element

like drivers' response time in traffic

- How do we actually model heat avalanche 'jam' ??? → origin in dynamics?

N.B. Barenblatt first proposed relation of time delay to layering

Traffic jam dynamics: ‘jamiton’

- A model for Traffic jam dynamics → Whitham

$$\rho_t + (\rho v)_x = 0$$

$$v_t + v v_x = -\frac{1}{\tau}\left\{ v - V(\rho) + \frac{\nu}{\rho}\rho_x \right\}$$

ρ → car density

v → traffic flow velocity

$V(\rho) - \frac{\nu}{\rho}\rho_x$ → an equilibrium traffic flow

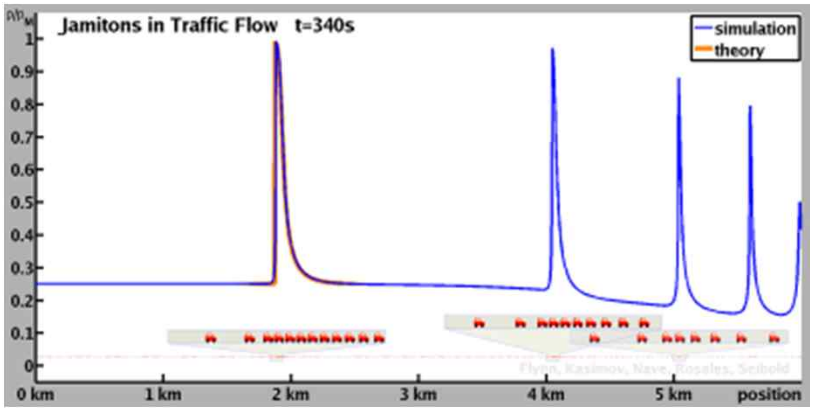
τ → driver’ s response time

→ Instability occurs when $\tau > \nu/(\rho_0^2 V_0'^2)$

$$D_{eff} = \nu - \tau\rho_0^2 V_0'^2 < 0$$ → clustering instability

→ Indicative of jam formation

- Simulation of traffic jam formation



http://math.mit.edu/projects/traffic/

→ Jamitons (Flynn, et.al., ’ 08)

n.b. I.V.P. → decay study

Heat avalanche dynamics model (`the usual')

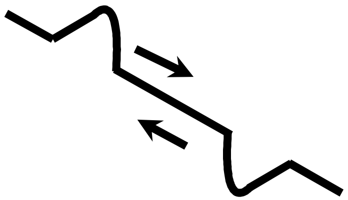
Hwa+Kardar ' 92, P.D. + Hahm ' 95, Carreras, et al. ' 96, ... GK simulation, ... Dif-Pradalier ' 10

- δT :deviation from marginal profile → conserved order parameter

- Heat Balance Eq.: $\partial_t \delta T + \partial_x Q[\delta T] = 0$ → up to source and noise

- Heat Flux $Q[\delta T]$? → utilize symmetry argument, ala' Ginzburg-Landau

  - Usual: → joint reflectional symmetry (Hwa+Kardar'92, Diamond+Hahm '95)



$$\delta T \leftrightarrow -\delta T$$
$$x \leftrightarrow -x$$

$$Q = Q_0(\delta T) = \frac{\lambda}{2}\delta T^2 - \chi_2 \partial_x \delta T + \chi_4 \partial_x^3 \delta T$$

↘ hyperdiffusion

lowest order → Burgers equation

$$\partial_t \delta T + \lambda \delta T \partial_x \delta T = \chi_2 \partial_x^2 \delta T$$

An extension of the heat avalanche dynamics

- An extension: a finite time of relaxation of Q toward SOC flux state

$$\partial_t Q = -\frac{1}{\tau}(Q - Q_0(\delta T))$$

(Guyot-Krumhansl)

$$Q_0[\delta T] = \frac{\lambda}{2}\delta T^2 - \chi_2 \partial_x \delta T + \chi_4 \partial_x^3 \delta T$$

→ In principle $\tau(\delta T, Q_0)$ ⟷ large near criticality (∼ critical slowing down)

i.e. enforces time delay between δT and heat flux

Soften flux–gradient relation

N.B.: Contrast quasi-linear theory!

- Dynamics of heat avalanche:

$$\partial_t \delta T + \lambda \delta T \partial_x \delta T = \chi_2 \partial_x^2 \delta T - \chi_4 \partial_x^4 \delta T - \tau \partial_t^2 \delta T$$

→ Burgers
(P.D. + T.S.H. '95)

New: finite response time

→ Telegraph equation

n.b. model for heat evolution

diffusion → Burgers → Telegraph

Relaxation time: the idea

- What is ' τ ' physically? → Learn from traffic jam dynamics
- A useful analogy:

heat avalanche dynamics	traffic flow dynamics
temp. deviation from marginal profile	local car density
heat flux	traffic flow
mean SOC flux (ala joint relflection symmetry)	equilibrium, steady traffic flow
heat flux relaxation time	driver's response time

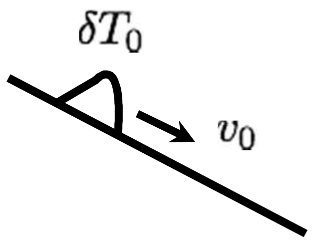
- driver's response can induce traffic jam

- jam in avalanche → profile corrugation → staircase?!?

- Key: instantaneous flux vs. mean flux

Analysis of heat avalanche dynamics via telegraph

- How do heat avalanches jam?



- Consider an initial avalanche,
  with amplitude δT_0,
  propagating at the speed $v_0 = \lambda \delta T_0$

→ turbulence model dependent

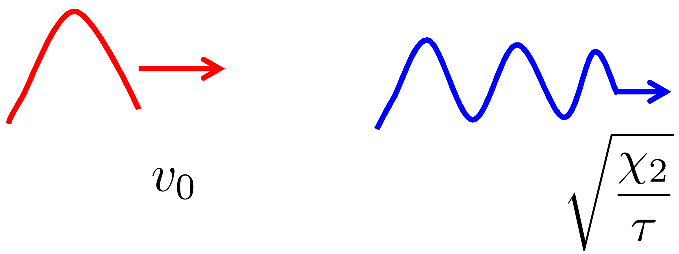
- Dynamics:

$$\partial_t \widetilde{\delta T} + v_0 \partial_x \widetilde{\delta T} = \chi_2 \partial_x^2 \widetilde{\delta T} - \chi_4 \partial_x^4 \widetilde{\delta T} - \tau \partial_t^2 \widetilde{\delta T}$$

pulse

'Heat flux wave': $\sqrt{\frac{\chi_2}{\tau}}$
telegraph → wavy feature

two characteristic propagation speeds



→ In short response time (usual)
heat flux wave propagates faster

→ In long response time, heat flux wave becomes slower and pulse starts overtaking.
What happens???

Analysis of heat avalanche jam dynamics

- In large tau limit, what happens? → Heat flux jams!!

- Recall plasma response time akin to driver's response time in traffic dynamics

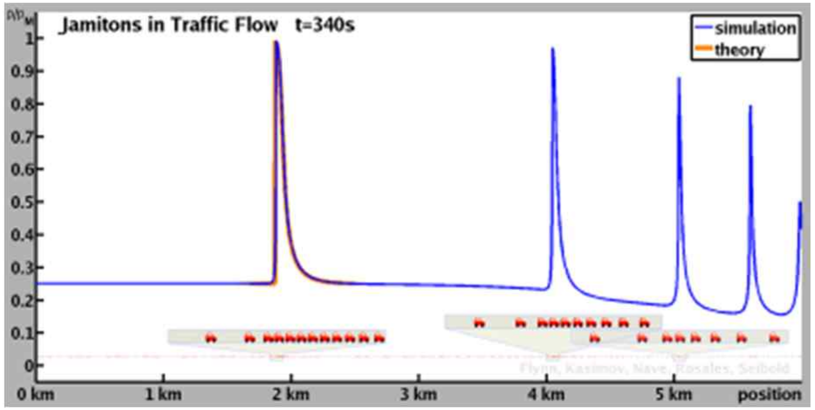
- negative heat conduction instability occurs (as in clustering instability in traffic jam dynamics)

$$\partial_t \widetilde{\delta T} + v_0 \partial_x \widetilde{\delta T} = \chi_2 \partial_x^2 \widetilde{\delta T} - \chi_4 \partial_x^4 \widetilde{\delta T} - \tau \partial_t^2 \widetilde{\delta T}$$

$$\rightarrow \underline{(\chi_2 - v_0^2 \tau)} \partial_x^2 \widetilde{\delta T} - \chi_4 \partial_x^4 \widetilde{\delta T}$$

<0 when overtaking

→ clustering instability



n.b. akin to negative viscosity instability of ZF in DW turbulence

instead ZF as secondary mode in the gas of primary DW

➔ Heat flux 'jamiton' as secondary mode in the gas of primary avalanches

Analysis of heat avalanche jam dynamics

- Growth rate of the jamiton instability

$$\gamma = -\frac{1}{2\tau} + \frac{1}{2\tau}\sqrt{\frac{r+1}{2} - 2\tau\chi_2 k^2\left(1+\frac{\chi_4 k^2}{\chi_2}\right)} \qquad r = \sqrt{\left\{4\tau\chi_2 k^2\left(1+\frac{\chi_4 k^2}{\chi_2}\right)-1\right\}^2 + 16 v_0^2 k^2 \tau^2}$$

- Threshold for instability

$$\tau > \frac{\chi_2}{v_0^2}\left(1+\frac{\chi_4 k^2}{\chi_2}\right)$$

→ critical minimal delay time

n.b. $1/\tau = 1/\tau[\mathcal{E}]$

→ clustering instability strongest near criticality

- Scale for maximum growth

$$k^2 \cong \frac{\chi_2}{\chi_4}\sqrt{\frac{\chi_4 v_0^2}{4\chi_2^3}}$$

from

$$\frac{\partial\gamma}{\partial k^2} = 0 \quad \Rightarrow \quad 8\tau\frac{\chi_4^2}{\chi_2}k^6 + 4\tau\chi_4 k^4 + 2\frac{\chi_4}{\chi_2}k^2 + 1 - \frac{v_0^2\tau}{\chi_2} = 0$$

→ staircase size, $\Delta^2_{stair}(\delta T)$, δT from saturation: consider shearing

Scaling of characteristic jam scale

- Saturation: Shearing strength to suppress clustering instability

Jam growth → profile corrugation → ExB staircase → $v'_{E\times B}$

→ estimate, only

→ saturated amplitude: $\frac{\delta T}{T_i} \sim \frac{1}{v_{thi}\rho_i}\sqrt{\frac{\chi_4}{\tau}}$

- Characteristic scale

$$\Delta^2 \sim k^{-2}(\delta T) \sim \frac{2v_{thi}}{\lambda T_i}\rho_i\sqrt{\chi_2\tau} \qquad \chi_2 \sim \chi_{neo}$$

- Geometric mean of ρ_i and $\sqrt{\chi_2\tau}$: ambient diffusion length in 1 relaxation time

- 'standard' parameters: $\Delta \sim 10\Delta_c$

Discussion

- "Negative diffusion" / clustering instability common to both Phillips and Jam mechanisms

Phillips → $\delta\Gamma/\delta\nabla b < 0$ → Γ nonlinearity

Jam → $\chi_2 - V_0^2\,\tau_{delay} < 0$ → τ_d physics

∴ Negative diffusion a general staircase forming mechanism

➔ Staircase formation is a generic form of secondary pattern instability in gradient-driven turbulence. Should be treated on equal footing with zonal flow, streamer, …

Fluctuation intensity profile of great interest

Discussion

- Similar to familiar transport bifurcation in $\delta\Gamma/\delta\nabla b < 0$
- Different in no "second state" supported by collisional transport
- Sets step width via turbulence spreading
- More general than $V'_{E\times B}$ suppression scenario
- Jam mechanism:
  - τ_d is key quantity
  - how do nonlinear couplings scatter flux? is central question

Re: Relaxation

- Relaxation theories generally predict “smooth” states
- Instructive to look at selective decay constraint on flux
- These can be modulationally unstable to staircases, etc.
- Is actual final state determined by structural merger process?
- Prediction ?! – barrier location?
- General issue is type of nonlinear process in play:
  - Cascading
  - Modulational instability
  - Bubble competition

c.f. Mcwilliams ‘84
- Stage 1 : cascading
- Stage 2 : structure interaction

➔ Is relaxation a multi-state process ??

Open Questions

- Staircase structure with spreading and residual diffusion?
- Staircase structure in inhomogeneous system → meso-micro interaction, profiled forcing?
- Multi-field staircase model (cf. experience with transport bifurcation → difficult!)
- Propagating solutions key: transit vs merger rate
- Noise effects (i.e. non-stationary forcing in time)
- Net flux drive

→ Is relaxation a multi-stage process? Characterization?

Approach ?!

- Staircase solutions require self-consistent treatment of gradient

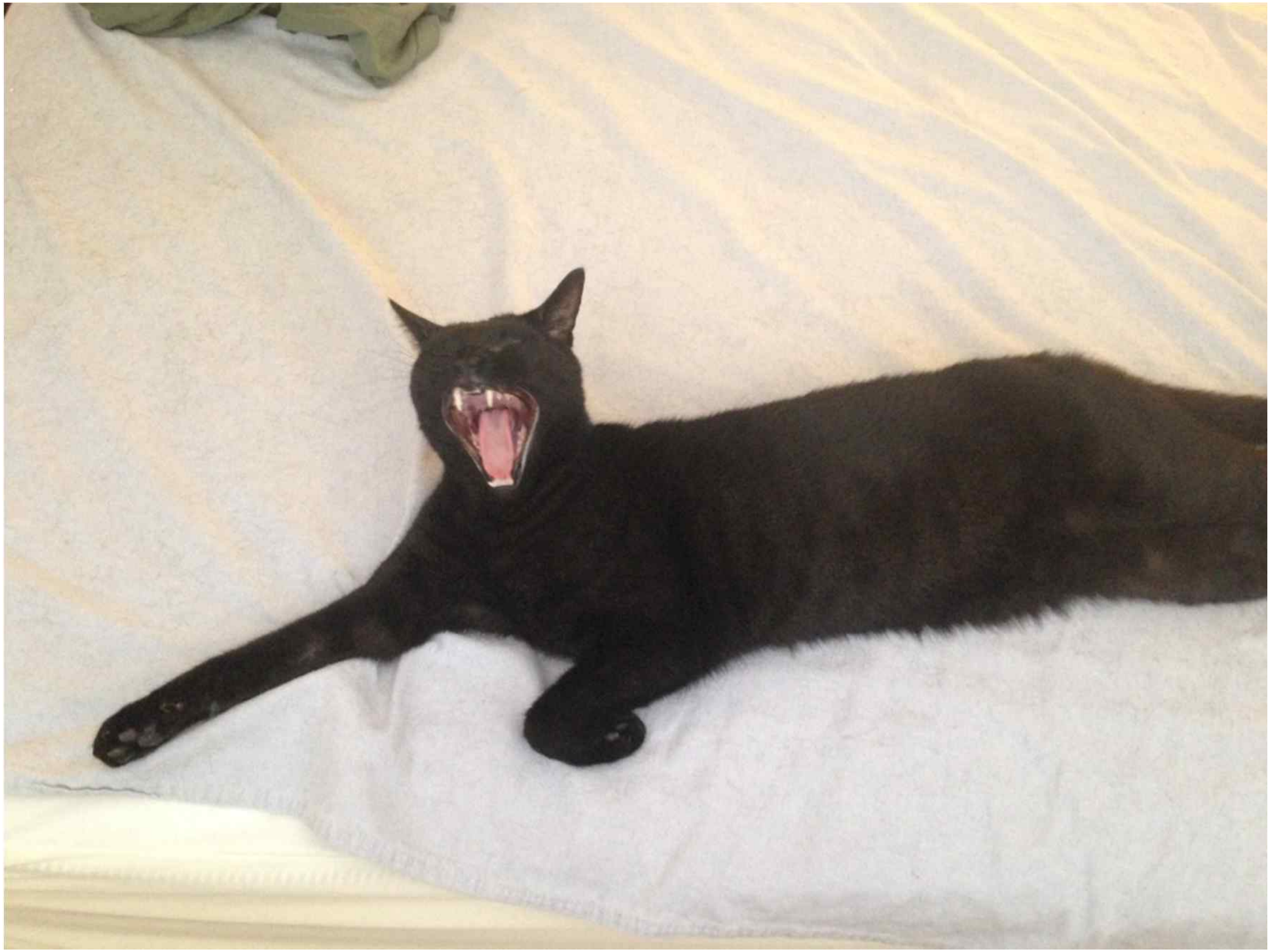
But

- GK, full toroidal geometry etc. all seem overkill and unnecessary to explore fundamentals; pain/gain $\rightarrow \infty$
- Especially important to 'turn down' neoclassical transport, collisional flow damping to reveal strong nonlinearity

So

- Simplify model:
  - Darmet ?
  - Flux driven fluid models ?!

  N.B. These have performed well in transport bifurcation studies



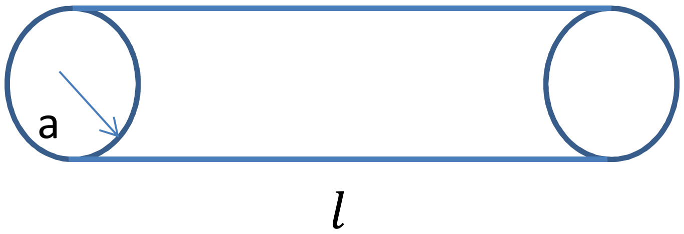
Bring on the prey…

Back-Up

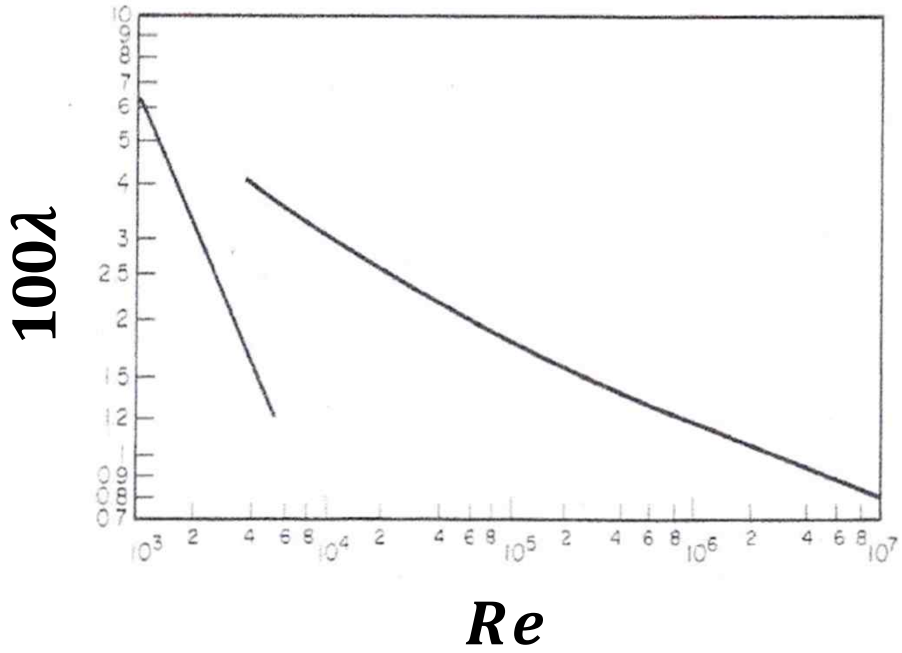
A Simpler (?!) Problem:
→ Turbulent Pipe Flow

- Essence of confinement:
  - given device, sources; what profile is achieved?
  - $\tau_E = W/P_{in}$
- Related problem: Pipe flow (turbulent)

ΔP → pressure drop



$$\Delta P \pi a^2 = \rho V_*^2 2\pi a l$$

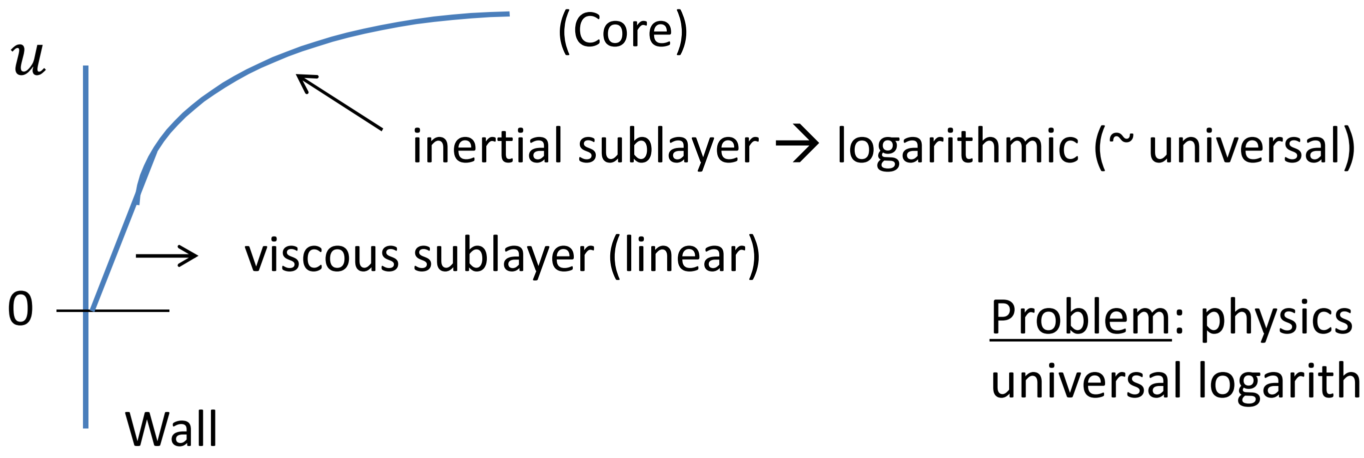


Balance: momentum transport to wall

(Reynolds stress) vs ΔP

➔ Flow profile

$$\lambda = \frac{2a\Delta P/l}{1/2\rho u^2}$$



<u>Problem</u>: physics of ~ universal logarithmic profile?

- <u>Prandtl Mixing Length Theory</u> (1932)
  - Wall stress = $\rho V_*^2 = -\rho \nu_T \, \partial u / \partial x$

    ↖ eddy viscosity

  - Absence of characteristic scale →

$$\nu_T \sim V_* x$$

$$u \sim V_* \ln(x/x_0)$$

$x \equiv$ mixing length, distance from wall

Analogy with kinetic theory …

$\nu_T = \nu \rightarrow x_0$, viscous layer → $x_0 = \nu / V_*$

Some key elements:

- Momentum flux driven process
- Turbulent diffusion model of transport eddy viscosity
- Mixing length:

  ~ x → macroscopic, eddys span system

  → ~ flat profile

- Self-similarity in radius
- Cut-off when $\nu_T = \nu$
- Reduce drag by creation of buffer layer i.e. steeper gradient than inertial sublayer (by polymer)

Structural MFT:

- The question of Dynamics brings us to mean field theory (c.f. Moffat '78 and an infinity of others)

- Mean Field Theory → how represent $\langle \tilde{v} \times \tilde{B} \rangle$?

  → how relate to relaxation ?

- Caveat: Perturbative MFT assumes fluctuations are small and quasi-Gaussian. They are often NOT

- Structural Approach (Boozer): (plasma frame)

$$\langle \mathbf{E} \rangle = \eta \langle \mathbf{J} \rangle + \langle \mathbf{S} \rangle$$ → 'something'

$\langle \mathbf{S} \rangle$ conserves H_M

$\langle \mathbf{S} \rangle$ dissipates E_M

Note this is ad-hoc, forcing $\langle \mathbf{S} \rangle$ to fit the conjecture. Not systematic.

Now

$$\partial_t H_M = -2c\eta \int d^3x \langle \mathbf{J} \cdot \mathbf{B} \rangle - 2c \int d^3x \langle \mathbf{S} \cdot \mathbf{B} \rangle$$

$$\therefore \langle \mathbf{S} \rangle = \frac{\mathbf{B}}{B^2} \nabla \cdot \mathbf{\Gamma}_H$$

→ Helicity flux

$$\partial_t \int d^3x \frac{B^2}{8\pi} = -\int d^3x \left[\eta J^2 - \mathbf{\Gamma}_H \cdot \nabla \frac{\langle \mathbf{J} \rangle \cdot \mathbf{B}}{B^2} \right]$$

so

$$\mathbf{\Gamma}_H = -\lambda \nabla (J_{\|}/B)$$

→ simplest form consistent with Taylor

→ turbulent hyper-resistivity $\lambda = \lambda[\langle \tilde{B}^2 \rangle]$ - ‘parameter’

→ Relaxed state: $\nabla (J_{\|}/B) \to 0$ homogenized current